A General Construction for PMDS Codes

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***Abstract*—PMDS (a.k.a. maximally recoverable) codes allow for local erasure recovery by utilizing row-wise parities and additional erasure correction through global parities. Recent works on PMDS codes focus on special case parameter settings, and a general construction for PMDS codes is stated as an open problem. This paper provides an explicit construction for PMDS codes for all parameters utilizing concatenation of Gabidulin and MDS codes, a technique originally proposed by Rawat et al. for constructing optimal locally repairable codes. This approach allows for PMDS constructions for any parameters albeit with large field sizes. To lower the field size, a relaxation on the rate requirement is considered, and PMDS codes based on combinatorial designs are constructed.**

I. Introduction

Redundant array of independent disks (RAID) [1] architecture is used to prevent systems from data loss in case of catastrophic failures (disk failure). Maximum distance separable (MDS) codes, i.e., Reed-Solomon codes, can be utilized for erasure correcting in RAID systems, i.e., in RAID 6 to overcome the failure of two disks. However, using solid-state drives (instead of hard disk drives) brought challenges, e.g., the system may experience both *disk failures* and *hard errors* which may not be realized unless the specific sector is accessed. RAID 6 architecture can tolerate such an erasure pattern. However, the cost of recovery is expensive. Partial MDS (PMDS) codes are proposed to overcome this problem by utilizing both row-wise parities and global parities to recover from mixed failures [2]. Remarkably, in the distributed storage context, these row-wise (local) parities reduce the communication cost of maintenance operations, and together with global parities, provide maximal fault tolerance [3].

PMDS codes tolerate mixed failures consisting of column failures (referring to a disk) in an $r \times n$ array and additional failures (sectors). Each row in the array forms an MDS code, i.e., each row forms a *local group* for erasure correction of up to m symbols. Considering $r \times n$ array over a finite field $\mathbb{F}$, PMDS codes' properties are [4]: Each row is an $[n, n-m, m+1]$ MDS code, and any m elements per row plus any additional s erasures in the array can be recovered. PMDS codes are labeled with $(m; s)$ and defined in [4] as follows.

Definition 1. *Let $\mathcal{C}$ be a linear $[rn, k]$ code over a field such that when codewords are taken row-wise as $r \times n$ arrays, each row belongs in an $[n, n-m, m+1]$ MDS code. $\mathcal{C}$ is an $(m; s)$ partial MDS (PMDS) code if, for any $(s_1, s_2, \dots, s_t)$ such that each $s_j \geq 1$ and $\sum_{j=1}^{t} s_j = s$, and for any $i_1, i_2, \dots, i_t$ such that $0 \leq i_1 < i_2 < \cdots < i_t \leq r-1$, $\mathcal{C}$ can correct up to $s_j + m$ erasures in each row i_j, $1 \leq j \leq t$.*

PMDS codes draw attention recently and code constructions are proposed in the literature, however, the parameter set ($(m; s)$ values) is limited. Explicit constructions are provided in [5], [6] for $(m; s) = (1; 1)$ and $(m; s) = (\leq 2; 2)$, in [2], [7] for $(m; s) = (\geq 1; 1)$, in [8] for $(m; s) = (\geq 1; 2)$, in [3] for $(m; s) = (1; 3)$ and $(m; s) = (1; 4)$, in [2] for $(m; s) = (1; \geq 1)$ and in [9] for $(m; s) = (\geq 1; 1)$. In all these explicit PMDS constructions, m or s is set to be 1 or 2.

Coding schemes that can be considered as relaxations to erasure recovery properties of PMDS codes include SD codes [4], STAIR codes [10], and t-level Generalized Concatenated (GC) codes [7]. Locally repairable codes (LRCs) has been studied recently [11]–[14], and these codes allow a recovery of a symbol within a corresponding *local group*. We remark that d_{min}-optimal LRCs necessarily have disjoint local groups, which make them as candidates for constructing PMDS codes. However, this approach (utilizing d_{min}-optimal LRCs) produces PMDS codes only for special parameter settings.

In this study, we first propose an explicit PMDS code construction for all parameters using concatenation of Gabidulin (Section II) and MDS codes, a technique originally proposed in [13] for constructing optimal LRCs. The general PMDS construction along with examples are detailed in Section III. Then, to lower the field size requirement of this approach, we develop rate suboptimal PMDS constructions using combinatorial designs in Section IV. In particular, we will refer to the PMDS definition given above as rate-optimal PMDS, where the corresponding rate is $R^* = \frac{r(n-m)-s}{rn}$, and compare this optimal rate with those of suboptimal rate codes.

II. Maximum Rank Distance (MRD) codes

We first define rank distance and linearized polynomials.

Definition 2 (Column rank)**.** *For a given basis of $\mathbb{F}_{q^M}$ over $\mathbb{F}_q$, the column rank of a vector $\mathbf{v} \in \mathbb{F}_{q^M}^N$ over the base field $\mathbb{F}_q$, denoted by $Rk(\mathbf{v}|\mathbb{F}_q)$, is the maximum number of linearly independent coordinates of $\mathbf{v}$ over the base field $\mathbb{F}_q$.*

We note that a basis establishes an isomorphism between N-length vectors, in $\mathbb{F}_{q^M}^N$, to $M \times N$ matrices, in $\mathbb{F}_q^{M \times N}$. For the given basis, the column rank $Rk(\mathbf{v}|\mathbb{F}_q)$ is equal to $rank(\mathbf{V})$, the rank of the corresponding matrix of $\mathbf{v}$.

Definition 3 (Rank distance)**.** *Rank distance between two vectors is defined by $d_R(\mathbf{v}_1, \mathbf{v}_2) = Rk(\mathbf{v}_1 - \mathbf{v}_2|\mathbb{F}_q)$.*

Definition 4 (Matrix (array) code)**.** *A matrix code is defined as any nonempty subset of $\mathbb{F}_q^{M \times N}$. (Also called array code [15].)*

Rank-metric code is a matrix (array) code, where the distance is the rank distance. The minimum distance of a rank-metric code $\mathcal{C} \subseteq \mathbb{F}_q^{M \times N}$ is given by

Manuscript received May 23, 2016; revised September 13, 2016; accepted October 26, 2016. Date of publication XXX XX, 2016; date of current version XXX XX, 2016. This work was supported by NSF under grant CCF-1563622. The associate editor coordinating the review of this paper and approving it for publication was C. Feng.

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Digital Object Identifier XX.XXXX//LCOMM.XXXX.XXXXXX

$$d_R(\mathcal{C}) = \min_{\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{C}; \mathbf{v}_1 \neq \mathbf{v}_2} d_R(\mathbf{v}_1, \mathbf{v}_2). \quad (1)$$

Lemma 5 ([16]). *Consider a rank-metric code $\mathcal{C} \subseteq \mathbb{F}_q^{M \times N}$ with minimum distance $d_R(\mathcal{C}) = D$. Then,*

$$\log_q(|\mathcal{C}|) \leq \min\{M(N-D+1), N(N-D+1)\}. \quad (2)$$

The codes that achieve the bound in (2) are called *maximum rank distance* (MRD) codes.

Definition 6. *A linearized polynomial $f(g)$ over $\mathbb{F}_{q^M}$ of q-degree $K-1$ has the form $f(g) = \sum_{i=0}^{K-1} c_i g^{[i]}$, where the coefficients $c_i \in \mathbb{F}_{q^M}$, $c_{K-1} \neq 0$, and $[i] = q^i$.*

We note that the linearized polynomial satisfies $f(a_1 g_1 + a_2 g_2) = a_1 f(g_1) + a_2 f(g_2)$, for $a_1, a_2 \in \mathbb{F}_q$ and $g_1, g_2 \in \mathbb{F}_{q^M}$. We provide Gabidulin construction of MRD codes [15], [16].

Definition 7 (MRD (Gabidulin) codes). *An $[N, K, D]$ MRD code $\mathcal{C}_{MRD}$ over the extension field $\mathbb{F}_{q^M}$ ($M \geq N$) has length-K input $u_0, \cdots, u_{K-1}$, where $u_i \in \mathbb{F}_{q^M}, i = 0, \cdots, K-1$, and encodes the input to length-N codewords by $x_j = f(g_j) = \sum_{i=0}^{K-1} u_i g_j^{[i]}$, for $j = 1, \cdots, N$. Linearized polynomial is evaluated with N linearly independent, over $\mathbb{F}_q$, generator elements $\{g_1, \cdots, g_N\}$ with $g_j \in \mathbb{F}_{q^M}$ which forms $\mathbf{G}_{MRD}$; and its coefficients are selected by the length-K input vector.*

Note that, symbol erasures in the vector representation of a codeword in the Gabidulin code correspond to column erasures in the matrix representation. Here, any non zero code (vector) has a rank (norm) of at least $N - K + 1$. Thus, the Gabidulin code achieves a rank distance of $D = N - K + 1$, which is the maximum achievable, and can correct any $D - 1$ erasures.

III. A GENERAL CONSTRUCTION FOR PMDS CODES

Recently, a concatenation of MRD and MDS array codes are utilized for coding in distributed storage systems. This approach is used for constructing LRCs in [13], LRCs with minimum bandwidth node repairs in [17], thwarting adversarial errors in [18], and secure cooperative regenerating codes in [19]. We utilize the same concatenation approach here to construct PMDS codes. We note that maximally recoverable codes in [3] are constructed using parity-check matrices and they also utilize the linearized polynomial property.

Construction I. [An $(m; s)$ PMDS code over an array of (r, n) symbols (r rows and n columns)] Set $K = r(n-m) - s$, and consider data symbols $\{u_0, \cdots, u_{K-1}\}$.

- Use $[N = K+s, K, D = s+1]$ Gabidulin code to encode $\{u_0, \cdots, u_{K-1}\}$ to length-N codeword $(x_1, \cdots, x_N)$. That is, $(x_1, \cdots, x_N) = (f(g_1), \cdots, f(g_N))$, where the linearized polynomial $f(g) = u_0 g^{[0]} + \cdots + u_{K-1} g^{[K-1]}$ is evaluated with N linearly independent, over $\mathbb{F}_q$, generator elements $\{g_1, \cdots, g_N\}$ each in $\mathbb{F}_{q^M}$; and its coefficients are selected by the length-K input vector. We represent this operation by writing $\mathbf{x} = \mathbf{u}\mathbf{G}_{\text{MRD}}$.
- Split resulting $N = K + s = r(n-m)$ symbols $\{x_1, \cdots, x_N\}$ into r rows each with n-m symbols. We represent this operation by double indexing the codeword symbols, i.e., $x_{i,j}$ is the symbol at row i and column j for $i = 1, \cdots, r$, $j = 1, \cdots, n-m$. We also denote the resulting sets with the vector notation, $\mathbf{x}_{i,1:n-m} = (x_{i,1}, x_{i,2}, \cdots, x_{i,n-m})$ for row i.
- Use an $[n, k = n-m, d = m+1]$ MDS array code for each row to construct additional parities. Representing the output symbols as $\mathbf{y}_{i,1:n}$ we have $\mathbf{y}_{i,1:n} = \mathbf{x}_{i,1:n-m}\mathbf{G}_{\text{MDS}}$ for each row i, where $\mathbf{G}_{\text{MDS}}$ is the encoding matrix of the MDS code over $\mathbb{F}_q$. If a systematic code is used, $\mathbf{x}_{i,1:n-m}$ is encoded into the vector $\mathbf{y}_{i,1:n} = (x_{i,1}, \cdots, x_{i,n-m}, p_{i,1}, \cdots, p_{i,m})$ for each row i.

The resulting codeword symbols are represented as a matrix:

$$\begin{array}{|cccc|} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{r,1} & y_{r,2} & \cdots & y_{r,n} \end{array} \quad (3)$$

Proposition 8. *The symbol matrix resulting from Construction 1 has a total of rn symbols that are placed in r rows and n columns. Now, consider that we have m erasures per row, and an additional s erasures over the remaining symbols (referred to as $(m; s)$ erasure pattern). The remaining symbols are sufficient to decode the data symbols $u_0, \cdots, u_{K-1}$, from which the erasures in $(m; s)$ erasure pattern can be recovered by re-encoding the data.*

Example 9. *Consider construction of $(m = 2; s = 3)$ PMDS over an array of $r = 3, n = 5$ symbols. Here, we use $[N = 9, K = 6, D = 4]$ Gabidulin code together with an $[n = 5, k = 3, d = 3]$ MDS code. We obtain, e.g., the following symbols, for the case of systematic MDS code.*

$$\begin{array}{|ccccc|} x_{1,1} & x_{1,2} & x_{1,3} & p_{1,1} & p_{1,2} \\ x_{2,1} & x_{2,2} & x_{2,3} & p_{2,1} & p_{2,2} \\ x_{3,1} & x_{3,2} & x_{3,3} & p_{3,1} & p_{3,2} \end{array} \quad (4)$$

(4) *can have $m = 2$ erasures in each row and additional $s = 3$ erasures. For instance, consider the erasures below.*

$$\begin{array}{|ccccc|} x_{1,1} & x_{1,2} & x_{1,3} & * & * \\ * & * & * & p_{2,1} & p_{2,2} \\ * & * & * & * & p_{3,2} \end{array} \quad (5)$$

This resulting symbol array $(x_{1,1}, x_{1,2}, x_{1,3}, p_{2,1}, p_{2,2}, p_{3,2})$ forms a set of linearly independent evaluation points of the underlying linearized polynomial for the $[N = 9, K = 6, D = 4]$ Gabidulin code. By polynomial interpolation, one can then solve for the data coefficients $u_0, \cdots, u_5$, re-encode this into codewords and construct back the full symbol matrix.

Proof of Proposition 8. We first provide a lemma, which is a summary of the observations given in [13] for the scenario considered here. (In particular, we have scalar symbols here.)

Lemma 10. *Consider the code given in Construction 1, where the Gabidulin codeword $\mathbf{x} = [x_1, \cdots, x_N] = [f(g_1), \cdots, f(g_N)]$ in $\mathbb{F}_{q^M}^N$ is partitioned into symbol vectors $\mathbf{x}_{i,1:n-m} = (x_{i,1}, \cdots, x_{i,n-m})$ for row $i = 1, \cdots, r$, and each is encoded into symbols $\mathbf{y}_{i,1:n}$ through G_{MDS}. Consider a set $\mathcal{S}$ which is the union of l_i symbols from row i (symbols in $\mathbf{y}_{i,1:n}$). Then, the symbols in $\mathcal{S}$ correspond to the evaluations of the underlying linearized polynomial $f(\cdot)$ at $\sum_{i=1}^{r} \min\{l_i, k\}$ linearly independent (over $\mathbb{F}_q$) points from $\mathbb{F}_{q^m}$.*

The proof of this lemma is provided in Appendix A. We utilize this Lemma in the following.

Corollary 11. *Consider the code given in Construction I and an erasure pattern which leaves l_i number of remaining symbols in row i in the symbol matrix. In such a scenario, if $\sum_{i=1}^{r} \min\{l_i, k\} \geq K$, then, the erasure pattern can be recovered from the remaining symbols. In particular, the remaining symbols result in $\sum_{i=1}^{r} \min\{l_i, k\}$ linearly independent evaluation points for the underlying polynomial (see Lemma 10). And, when this number is greater than or equal to K, the data symbols $u_0, \cdots, u_{K-1}$ can be decoded via polynomial interpolation, from which the pre-erasure situation of the array can be recovered by re-encoding the symbols.*

For a given $(m; s)$ erasure scenario over an array of (r, n) symbols (r rows and n columns), we have m erasures in each row and additional s_i erasures per row, resulting in a total of $rm + \sum_{i=1}^{r} s_i = rm + s$ erasures. In Construction 1, after erasing m symbols from each row, we are left with $n - m$ symbols in r rows. Now, having s_i number of additional erasures in each row will result in having $l_i = n - m - s_i$ number of symbols at row i. As the underlying MDS code has a dimension of $k = n - m$, the number of linearly independent evaluations at hand is $\sum_{i=1}^{r} \min\{l_i, k\} = \sum_{i=1}^{r} l_i = r(n-m) - \sum_{i=1}^{r} s_i = r(n-m) - s = K$. Therefore, any $(m; s)$ erasure pattern can be recovered with Construction 1. □

Remark 12. *Construction I is same as the one in [13]. We note that this construction, in addition to being an LRC, which provides row-wise MDS property of PMDS codes, has a maximum erasure tolerance property that matches to the $(m; s)$ erasure pattern recovery property of PMDS codes. Together with rate optimality of the construction, this provides a general construction for optimal PMDS codes for all parameters.*

Note that Construction I allows for construction of PMDS for any m and s, but with a field size of $q^{r(n-m)}$ ($M \geq N = r(n-m)$ from Definition 7), where $q \geq n$ due to $[n, k = n-m]$ MDS codes. On the other hand, the existing PMDS codes work for limited range of m or s (with lower field sizes). Next, we relax the optimal rate requirement in PMDS codes and provide constructions with lower field sizes.

IV. Rate suboptimal PMDS codes through combinatorial designs

We first provide an example. Assume a data $\mathcal{D} = \mathbf{u}$ of size 9 contains 3 sub-data ($\mathcal{D}_i = [u_{i,1:3}]$) each of size 3, i.e., $\mathbf{u} = \{u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}\}$. We encode each of these sub-data with $[10, 3]$ MDS code and represent the resulting elements with $\mathcal{P}_1 = p_{1,1:10}$ for $\mathcal{D}_1$, $\mathcal{P}_2 = p_{2,1:10}$ for $\mathcal{D}_2$, and $\mathcal{P}_3 = p_{3,1:10}$ for $\mathcal{D}_3$. We have

$$\begin{bmatrix} u_{i,1:3} \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_{i,1} & \alpha_{i,2} & \cdots & \alpha_{i,10} \\ \alpha_{i,1}^2 & \alpha_{i,2}^2 & \cdots & \alpha_{i,10}^2 \end{bmatrix} = \begin{bmatrix} p_{i,1:10} \end{bmatrix}. \quad (6)$$

These elements are grouped in a specific way placed into array as represented in Fig. 1 where each codeword symbol contains two elements each coming from two of the different sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$. Thus, each row now can be taken as $[5, 3]$ MDS code. Here, we can think of the generator matrix $\mathbf{G}$ of overall code $\mathcal{C}$ as consisting of 15 thick columns each of size 2 thin columns (corresponding to 2 different sub-data). Note that, the code can tolerate erasure of any $m = 2$ symbols per row plus any $s = 3$ symbols hence allowing recovery from PMDS erasure pattern since the remaining 12 elements (6 symbols) have at least 3 elements (3 thin columns) per sub-data from which each of the sub-data can be recovered and so is the original array. The general construction using a projective plane of order p is as follows.

	c_1	c_2	c_3	c_4	c_5
1	$p_{1,1}$ $p_{2,1}$	$p_{1,2}$ $p_{2,2}$	$p_{1,3}$ $p_{2,3}$	$p_{1,4}$ $p_{2,4}$	$p_{1,5}$ $p_{2,5}$
	c_6	c_7	c_8	c_9	c_{10}
2	$p_{3,1}$ $p_{1,6}$	$p_{3,2}$ $p_{1,7}$	$p_{3,3}$ $p_{1,8}$	$p_{3,4}$ $p_{1,9}$	$p_{3,5}$ $p_{1,10}$
	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
3	$p_{3,6}$ $p_{2,6}$	$p_{3,7}$ $p_{2,7}$	$p_{3,8}$ $p_{2,8}$	$p_{3,9}$ $p_{2,9}$	$p_{3,10}$ $p_{2,10}$

Fig. 1. MDS codewords corresponding to each sub-data are placed as symbols of the code according to the underlying projective plane.

Construction II. Assume we have a data $\mathcal{D}$ of size $r(n-m)$, and consider a projective plane of order p with PMDS parameters satisfying $(n-m)p = s$ and $r = p^2 + p + 1$. First, partition $\mathcal{D}$ into $r = p^2 + p + 1$ sub-data, where $p = \frac{s}{n-m}$. Then, encode each sub-data using $[n(p+1), n-m]$ MDS code and distribute the resulting $n(p+1)$ elements for each sub-data evenly to $p+1$ different rows (according to the underlying projective plane).

As a result of this construction, symbols in each row stores elements from $p+1$ distinct sub-data, hence a row can be considered as an $[n, n-m]$ MDS code since puncturing np coordinates from $[n(p+1), n-m]$ MDS code results in $[n, n-m]$ MDS code. We now show that erasure of any m symbols per row plus any s symbols can be tolerated.

Proof. Consider the generator matrix $\mathbf{G}$ which has r sub-block-matrix (corresponding to the rows), each having n thick columns (corresponding to symbols in each row). Each of these thick columns also have $p+1$ thin columns. Erasure of any m nodes per row is same as puncturing any m thick columns from each of the r sub-block-matrix. In addition, any s erasures corresponds to puncturing any additional s thick columns. Puncturing any m thick columns from each of the r sub-block-matrix has the same effect on each sub-data. However, the additional s erasures may have different effect on different sub-data depending on the erasure pattern. Since any two blocks in the projective plane has only one common point, any $s \geq 2$ thick columns contains at least one common sub-data. Considering the worst case of having all s punctured thick columns containing one common sub-data, the remaining thick columns contain at least $n(p+1) - m(p+1) - s$ thin columns for each of the sub-data. Since we have $p = \frac{s}{n-m}$ in the code construction, we have at least $n(p+1) - m(p+1) - p(n-m) = n - m$ thin columns for each of the sub-data. Therefore, using these $n-m$ thin columns, each of the sub-data can be decoded using the underlying MDS code and the original array can be reconstructed. □

Although this construction requires lower field size, $q \geq n(p+1)$, it is not rate optimal. The original data is of size $r(n-m)$ and storage cost is $rn(p+1)$ yielding rate as $R^{(II)} =$

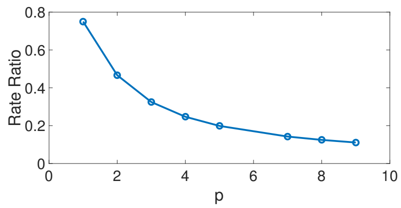

Fig. 2. Rate ratio of Construction II for various projective plane orders.

$\frac{n-m}{n(p+1)}$ and we have $\frac{R^{(II)}}{R^*} = \frac{p^2+p+1}{(p+1)(p^2+1)}$. For different values of p, we evaluate this ratio in Fig. 2. Note that for $p = 6$, there is no projective plane known. As the projective plane order increases, the rate ratio decreases and the required field size increases. One observation is that with projective plane construction, the system may tolerate even more than any s additional erasures (since construction is designed to tolerate the worst case of s). For example, using projective plane of order $p = 1$ for $(m = 2, r = 3, n = 5)$ we can tolerate $\%100$ of $s \leq 3$, $\%64.29$ of $s = 4$ and none of $s \geq 5$.

Construction III. Assume we have a data $\mathcal{D}$ and consider an $(v, \kappa, \lambda = 1)$-resolvable balanced incomplete design (RBIBD) satisfying $s = \frac{(n-m)(v-\kappa)}{\kappa-1}$ and $r = \frac{v(v-1)}{\kappa(\kappa-1)}$. First, partition $\mathcal{D}$ into v sub-data. Then, encode each sub-data using $[\frac{n(v-1)}{\kappa-1}, n-m]$ MDS code and distribute the resulting $\frac{n(v-1)}{\kappa-1}$ elements for each sub-data according to the underlying RBIBD.

A row in $r \times n$ array stores symbols from the same set of κ sub-data and since each sub-data is repeated $\frac{v-1}{\kappa-1}$ times, each row stores n symbols for each of the κ sub-data. That is, each row can be represented by a block of RBIBD. Any m erasures per row results in erasure of $\frac{m(v-1)}{\kappa-1}$ for each sub-data. Furthermore, assume the worst case that is the additional s erasures also occur involving a common sub-data, then at least $\frac{n(v-1)}{\kappa-1} - \frac{m(v-1)}{\kappa-1} - s$ symbols remain for each sub-data. Since $s = \frac{(n-m)(v-\kappa)}{\kappa-1}$, we have at least $n - m$ symbols for each sub-data, which is enough to decode each sub-data using the underlying MDS code and from which the original data can be decoded. Construction III yields rate suboptimal PMDS as $R^{(III)} = \frac{(n-m)(\kappa-1)}{n(v-1)}$ and we have $\frac{R^{(III)}}{R^*} = \frac{(\kappa-1)v}{v^2-v-v\kappa+\kappa^2}$. For example, using $(9, 3, 1)$-RBIBD results in $\frac{R^{(III)}}{R^*} = \frac{1}{3}$.

ACKNOWLEDGEMENT

The authors would like to thank Mario Blaum of IBM and Eitan Yaakobi of Technion for discussions on PMDS codes and for comments on the previous version of this manuscript.

APPENDIX

The proof follows from the linearized property of the polynomial utilized in the Gabidulin code. (The proof is given in Lemma 9 and 23 of [13] for the general case of having vector symbols. See also [17]. We provide a summary here for the scalar case.) Consider row i which encodes symbols $\mathbf{x}_{i,1:n-m}$ into $\mathbf{y}_{i:1:n} = \mathbf{x}_{i,n-m} G_{\text{MDS}}$. Here, representing the corresponding evaluation points with $g_{i,j}$ for row i, we have, as $k = n-m$, $\mathbf{x}_{i,1:k} = (f(g_{i,1}), f(g_{i,2}), \cdots, f(g_{i,k}))$. Now, denoting G_{MDS} as $k \times n$ matrix with entries $[G_{h,j}]$ for $h = 1, \cdots, k$ and $j = 1, \cdots, n$, we have $y_{i,j} = \sum_{h=1}^{k} f(g_{i,h}) G_{h,j}$. Due to the linearized property of $f(\cdot)$, we have $y_{i,j} = f(\sum_{h=1}^{k} G_{h,j} g_{i,h})$. Denote this new evaluation points as $\tilde{g}_{i,j} = \sum_{h=1}^{k} G_{h,j} g_{i,h}$. These points given by $\tilde{g}_{i,1:n}$ span the space spanned by the set $g_{i,1:k}$. Consider a set $\mathcal{S}_i \subseteq \{1, \cdots, n\}$ of size l_i. Due to the full rankness of the matrix G_{MDS}, the set of points $\{\tilde{g}_{i,j}, j \in \mathcal{S}_i\}$ span a $\min\{l_i, k\}$ dimensional space in the space spanned by $g_{i,1:k}$. Also, as the points in different rows, say $g_{i,1:k}$ and $g_{\tilde{i},1:k}$ for $i \neq \tilde{i}$, are independent, we have linear independence of $\tilde{g}_{i,1:n}$ and $\tilde{g}_{\tilde{i},1:n}$ for any $i \neq \tilde{i}$. Hence, the symbols in $\mathcal{S} = \cup_{i=1}^{r} \mathcal{S}_i$ correspond to the evaluations of the underlying linearized polynomial $f(\cdot)$ at $\sum_{i=1}^{r} \min\{l_i, k\}$ linearly independent (over $\mathbb{F}_q$) points from $\mathbb{F}_{q^m}$.

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