Polar Alignment for Interference Networks

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Abstract—Polar coding has originally been introduced as a capacity achieving low complexity code for binary input symmetric channels. Polar codes can be understood as transformations that replace a probabilistic channel with parallel deterministic counterparts. This paper builds on this interpretation of polar codes, using it to perform alignment over the resulting deterministic channels to obtain gains for interference networks. It is important to note here that polar codes are not chosen with encoding and decoding complexity in mind, which is just a fortuitous side-benefit, but with the aim of transforming the original channels into a class of deterministic parallel channels over which interference-alignment is well-understood. A degraded one-sided interference network is chosen as the illustrative example. Polar alignment is shown to increase the achievable sum rate over known random coding schemes. The paper concludes with a brief discussion of possible extensions.

I. Introduction

Polar codes, introduced by Arıkan [1], is a class of structured codes that are shown to be capacity achieving for discrete memoryless channels (DMCs). In its original form [1], "polarization" is tantamount to converting n uses of a given binary-input discrete memoryless channel (DMC) into n *polarized* channels. These polarized channels are deterministic channels which are either "good" or "bad", (i.e., either possessing a rate close to 1 or 0, respectively). Arıkan showed that the fraction of "good" channels converges to the symmetric capacity of the channel (which is the mutual information between the channel input and output given the input is chosen as a binary uniform random variable). Subsequently, a coding strategy which simply transmits information over the "good" channels achieve the symmetric capacity of the channel. (The input bits corresponding to bad channels are fixed, called "frozen", and shared with the receiver before the communication takes place in order to allow successive decoding at the receiver.) Remarkably, this technique achieves the capacity of any binary input symmetric DMC with low encoding and decoding complexity of $O(n \log n)$ and an exponential error probability decay of roughly $O(2^{-\sqrt{n}})$ for block length n. (The error decay rate is demonstrated in [2].)

Since its introduction, polar codes have been studied in multiple different contexts, which include a) systematic coding, non-binary inputs, and combinations with other coding strategies [3]–[13], b) applications to source coding [14]–[17], c) applications to the additive white Gaussian noise (AWGN) channel [8], [18], d) generating security [19]–[22],

e) and finally, in the context of multiple-access channels (MACs) [23], [24]. This last class of polar coding for MACs is particularly important for the application of polar codes in interference networks, as studied here.

This paper considers polar codes in the context of interference networks, with an aim of combining the concepts of interference-alignment [25], [26] with polar coding, a combination we call polar alignment. The 2-user interference channel was studied in [27]; and the best known achievable rate region for the two user scenario is given by Han and Kobayashi [28], is recently simplified in [29]. However, except for some special cases (e.g., [30]–[34]), characterizing the capacity region of the two-user Gaussian interference channel remains as an open problem. Recent works on interference channels having more than two users emphasize the importance of interference alignment in increasing the achievable rate regions for interference networks. Once aligned, the interference can be considered as noise [25] or can be decoded at the receivers [35]–[37]. Alignment, as studied in the vast and growing literature on the topic is based on structured coding schemes, and thus polar codes are a natural choice for an implementation of interference alignment.

The main contribution of this paper is the insight that polarization is a transformation that replaces the original (noisy) channel coding problem with multiple parallel deterministic channel coding problems. Interference-alignment is best understood in the context of such deterministic channels; and enabling alignment over general DMCs has proven to be a challenging task. Thus, polarization and alignment *naturally* work together to enable the characterization of good achievable regions for DMCs. Note that, such an approach not only enjoys the low complexity of encoding and decoding of polar codes and its deterministic coding and decoding structures, but also, perhaps more importantly, take advantage of the polarization in alignment of the interference.

The authors believe that a substantial body of work needs to be conducted to fully understand polar alignment for arbitrary interference networks. To provide a meaningful starting point, we initiate this study with the classical one-sided discrete memoryless 3-user interference channel with a degraded receiver structure. Achievable rates for 3-users interference channels has already been studied in existing literature using structured codes [35]–[37]. The added benefits of studying it from a polar coding context is to show that a simultaneous

polarization and alignment (of interference) can be achieved.

A detailed description of the one-sided interference channel studied in this paper is provided by the system model presented in the next section. In this paper, we combine the results on polar coding a q-ary input point-to-point and multiple-access channels [8], [23], [24] to show that it is possible to extend some known random coding inner bounds (such as the extension of the Han-Kobayashi coding [28] and the scheme of Bandemer and El Gamal [38]). The underlying idea is to decode (part of the) *sum of the interfering codewords* at the receiver that receives interference. We also discuss how the polar coding might be useful for alignment and decoding of the interference for general networks.

The rest of the paper is organized as follows. We introduce the system model in Section II, and describe two random coding inner bounds to the capacity region in Section III. In Section IV, we introduce the coding scheme and compare it with the rate regions obtained using random coding. In Section V, we comment on our findings and discuss how polar coding might find use in general interference networks, and, in Section VI, we conclude the paper.

II. SYSTEM MODEL

The discrete memoryless interference channel with 3-users is governed by $p(y_1, y_2, y_3|x_1, x_2, x_3)$, where $(x_1, x_2, x_3) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ are channel inputs, and $(y_1, y_2, y_3) \in \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3$ are the channel outputs. The channel is memoryless in the sense that the n usages of the channel can be written as

$$p(Y_1^n = y_1^n, Y_2^n = y_2^n, Y_3^n = y_3^n|$$
$$X_1^n = x_1^n, X_2^n = x_2^n, X_3^n = x_3^n)$$
$$= \prod_{i=1}^{n} p(y_1(i), y_2(i), y_3(i)|x_1(i), x_2(i), x_3(i)). \quad (1)$$

In this paper, we consider the special case of one sided interference channels over finite fields, where

$$\mathcal{X}_k = \mathcal{Y}_k = \{0, \cdots, q - 1\}, \text{ for } k = 1, 2, 3,$$

and the channels are given by

$$Y_1 = X_1 + aX_2 + aX_3 + N_1 \quad (2)$$
$$Y_2 = X_2 + N_2 \quad (3)$$
$$Y_3 = X_3 + N_3, \quad (4)$$

where N_k is an additive noise variable defined over GF(q) for $k = 1, 2, 3$, $a \in$ GF(q) is a fixed amplification factor, and the summations are over the field GF(q). The channel can be represented as shown in Figure 1. (Results provided in this paper can be easily extended to a model with distinct non-zero coefficients multiplying each factor in (2), (3), and (4).)

For our polar coding scheme, we focus on the "degraded" interference channels, formalized by the following Markov chain structure: Let

$$\tilde{Y}_1 \triangleq \tilde{X} + N_1 \quad (5)$$
$$\tilde{Y}_2 \triangleq \tilde{X} + N_2 \quad (6)$$
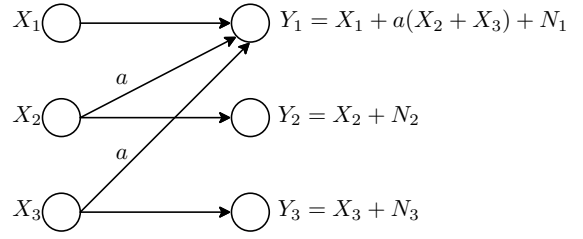$$\tilde{Y}_3 \triangleq \tilde{X} + N_3, \quad (7)$$

Fig. 1. The q-ary one-sided interference channel

then the channel is said to be degraded if $\tilde{X} \to \tilde{Y}_1 \to \tilde{Y}_2 \to \tilde{Y}_3$ forms a Markov chain. We observe that this condition also implies a strong interference situation, since we assume that the the interference $a(X_2 + X_3)$ is decodable at the first receiver whenever the second and third receiver are able to decode X_2 and X_3, respectively.

We denote the rate of user k with R_k, and a $(R_1, R_2, R_3, n, P_e^{(n)})$ code has the following components:

1) The message set of user k, $\mathcal{W}_k = \{1, \cdots, 2^{nR_k}\}$,

2) The encoding function at transmitter k, $f : \mathcal{W}_k \to \mathcal{X}_k^n$, mapping the messages to channel inputs,

3) The decoding function at receiver k, $g : \mathcal{Y}_k^n \to \mathcal{W}_k$, mapping the channel outputs to the estimates of the transmitted message, and

4) The average error probability $P_e^{(n)}$, which is given by

$$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2+R_3)}} \sum_{w_k \in \mathcal{W}_k, k=1,2,3} \Pr\{g(Y_1^n) \neq w_1 \text{ or}$$
$$g(Y_2^n) \neq w_2 \text{ or } g(Y_3^n) \neq w_3 | (w_1, w_2, w_3) \text{ is sent.}\} \quad (8)$$

The tuple (R_1, R_2, R_3) is said to be achievable, if there exist a sequence of $(R_1, R_2, R_3, n, P_e^{(n)})$ codes satisfying arbitrarily small error probability in the limit of large n (i.e., codes with $P_e^{(n)} \to 0$ as $n \to \infty$). The capacity region is the set of all achievable (R_1, R_2, R_3) pairs and is denoted by $\mathcal{C}$.

All logarithms are taken with respect to base q in this paper, where q is a prime number.

III. RANDOM CODING SCHEMES

A. An extension of the Han-Kobayashi scheme

The Han-Kobayashi scheme [28] for the interference channels can be extended to the 3-user scenario; where, for the one-sided model, the users will have the following splitting over the messages: $W_1 = W_{1p}$, $W_2 = \{W_{2c}, W_{2p}\}$, $W_3 = \{W_{3c}, W_{3p}\}$, where W_{kp} is the private message decoded at receiver k for $k = 1, 2, 3$, and W_{kc} is the common message decoded at both receivers k and 1 for $k = 1, 2$. We denote the corresponding rates of the messages with R_{kp} and R_{kc}, and denote the codewords carrying common and private messages with U_k^n and V_k^n, respectively, for $k = 1, 2$, where $(U_k, V_k) \in \mathcal{U}_k \times \mathcal{V}_k$ for some finite sets $\mathcal{U}_k$ and $\mathcal{V}_k$. (We directly use X_1^n for W_1.) We also define the time-sharing random variable Q over the finite set $\mathcal{Q}$. Let $\mathcal{P}_1$ be the set

of input distributions that factors as

$$p(q, x_1, u_2, v_2, x_2, u_3, v_3, x_3) =$$
$$p(q)p(u_2|q)p(v_2|q)p(u_3|q)p(v_3|q)$$
$$p(x_1|q)p(x_2|u_2, v_2, q)p(x_3|u_3, v_3, q). \quad (9)$$

Then, the following region is achievable.

Corollary 1 (HK Scheme [28]): For a given input distribution $p \in \mathcal{P}_1$, let $\mathcal{R}_1(p)$ is the set of all non-negative $(R_1, R_{2c}, R_{2p}, R_{3c}, R_{3p})$ tuples satisfying the following inequalities.

$$
\begin{aligned}
R_1 &\leq I(X_1; Y_1|U_2, U_3, Q) \\
R_{2c} &\leq I(U_2; Y_1|X_1, U_3, Q) \\
R_{3c} &\leq I(U_3; Y_1|X_1, U_2, Q) \\
R_1 + R_{2c} &\leq I(X_1, U_2; Y_1|U_3, Q) \\
R_1 + R_{3c} &\leq I(X_1, U_3; Y_1|U_2, Q) \\
R_{2c} + R_{3c} &\leq I(U_2, U_3; Y_1|X_1, Q) \\
R_1 + R_{2c} + R_{3c} &\leq I(X_1, U_2, U_3; Y_1|Q) \\
R_{2c} &\leq I(U_2; Y_2|V_2, Q) \\
R_{2p} &\leq I(V_2; Y_2|U_2, Q) \\
R_{2c} + R_{2p} &\leq I(U_2, V_2; Y_2|Q) \\
R_{3c} &\leq I(U_3; Y_3|V_3, Q) \\
R_{3p} &\leq I(V_3; Y_3|U_3, Q) \\
R_{3c} + R_{3p} &\leq I(U_3, V_3; Y_3|Q)
\end{aligned}
$$

Let $\pi(\mathcal{S})$ be the set of (R_1, R_2, R_3) such that $R_2 = R_{2c} + R_{2p}$ and $R_3 = R_{3c} + R_{3p}$ for some set $\mathcal{S}$ consisting of $(R_1, R_{2c}, R_{2p}, R_{3c}, R_{3p})$ tuples. Then,

$$R_{\text{HK}} \triangleq \pi\left(\cup_{p \in \mathcal{P}_1} \mathcal{R}_1(p)\right) \subseteq \mathcal{C}. \quad (10)$$

B. Decoding of the combined interference

Interference decoding with random codes is proposed in [38] for 3-user interference channel. When specialized to the one-sided model given in the previous section, this scheme achieves the following region.

Corollary 2 (Interference decoding scheme [38]): Let $\mathcal{P}_2$ be the set of input distributions that factors as

$$p(q, x_1, x_2, x_3) = p(q)p(x_1|q)p(x_2|q)p(x_3|q). \quad (11)$$

For a given input distribution $p \in \mathcal{P}_2$, let $\mathcal{R}_2(p)$ is the set of all non-negative (R_1, R_2, R_3) tuples satisfying the following inequalities.

$$
\begin{aligned}
R_1 &< I(X_1; Y_1|a(X_2 + X_3), Q) \\
R_1 + \min\{R_2, H(X_2|Q)\} &< I(X_1, X_2; Y_1|X_3, Q) \\
R_1 + \min\{R_3, H(X_3|Q)\} &< I(X_1, X_3; Y_1|X_2, Q) \\
R_2 &< I(X_2; Y_2|Q) \\
R_3 &< I(X_3; Y_3|Q),
\end{aligned}
$$

and

$$R_1 + \min\{R_2 + R_3, R_2 + H(X_3|Q), R_3 + H(X_2|Q),$$
$$H(a(X_2 + X_3))\} < I(X_1, a(X_2 + X_3); Y_1|Q). \quad (12)$$

Then,

$$R_{\text{ID}} \triangleq \cup_{p \in \mathcal{P}_2} \mathcal{R}_2(p) \subseteq \mathcal{C}. \quad (13)$$

IV. POLAR CODING FOR ONE-SIDED INTERFERENCE CHANNELS

In this section, we shall describe a polar coding based achievability strategy for the one-sided interference channel with degraded receivers. We show that appropriate use of polar codes on a one-sided degraded interference channel can enhance the achievable rate region, and in particular increase the achievable sum-rate.

A. Polar Codes for the q-ary MAC Channel

In this section, we provide some definitions which faciliate the use of Polar codes for the q-input q-ary output MAC. (Please refer to [23], [24] for details.) We assume that the input and output alphabets are $\mathcal{X} = \mathcal{Y} = \{0, 1, \ldots q - 1\}$ and that all logarithms are to the base q. The 2-user MAC is specified by the conditional probabilities as:

$$P(y|\bar{x}), \text{ for each } y \in \mathcal{Y} \text{ and}$$
$$\bar{x} = (x_1, x_2) \in GF(q) \times GF(q)$$

Let $E_2 \triangleq \{1, 2\}$ and $X_1(i), X_2(i)$ represent the mutually independent and identically distributed q-ary random variables transmitted by each user at the i-th time instant. Let $\bar{X}(i) \triangleq (X_1(i), X_2(i))$. Let the output of the MAC P for input $\bar{X}(i)$ be $Y(i)$. For a set $J \subseteq E_2$, we define

$$
\begin{aligned}
X_J &\triangleq \{X_k : k \in J\} \\
I_J(P) &\triangleq I(X_J; YX_{J^C})
\end{aligned}
$$

where $J^C = E_2 - J$.

We are interested in polarization of this MAC when the same construction of Arıkan is used. It is shown in [23] and [24] that the two-user MAC has five extremal channels (compared to the two extremals of the point-to-point channel). We here briefly describe this phenomenon. (Please refer to [23] and [24] for details.)

Upon two successive independent uses of the channel, with respective input $\bar{X}(1) = (X_1(1), X_2(1))$ and $\bar{X}(2) = (X_1(2), X_2(2))$, we obtain the respective outputs $Y(1)$ and $Y(2)$:

$$\bar{X}(1) \to Y(1), \quad \bar{X}(2) \to Y(2)$$

To completely characterize the achievable rate, we define two auxiliary q-ary random vectors

$$
\begin{aligned}
\bar{U}(1) &\triangleq (U_1(1), U_2(1)) \\
\bar{U}(2) &\triangleq (U_1(2), U_2(2))
\end{aligned}
$$

which are mutually independent and consist of uniformly distributed components. We then connect these with $\bar{X}(1)$ and $\bar{X}(2)$ as

$$
\begin{aligned}
\bar{X}(1) &= \bar{U}(1) + \bar{U}(2) \\
\bar{X}(2) &= \bar{U}(2),
\end{aligned}
$$

where addition is defined component-wise modulo q.

Definition 3: Let $P : \text{GF}(q) \times \text{GF}(q) \rightarrow \mathcal{Y}$ be an 2-user q-ary MAC. Let $P^- : \text{GF}(q) \times \text{GF}(q) \rightarrow \mathcal{Y}^2$ and $P^+ : \text{GF}(q) \times \text{GF}(q) \rightarrow \mathcal{Y}^2$ be two new MACs given by

$$P(y(1), y(2) | \bar{u}(1)) \triangleq$$

$$\sum_{\bar{u}(2)} \frac{1}{q^2} P(y(1) | \bar{u}(1) + \bar{u}(2)) P(y(2) | \bar{u}(2)),$$

$$P(y(1), y(2), \bar{u}(1) | \bar{u}(2)) \triangleq$$

$$\frac{1}{q^2} P(y(1) | \bar{u}(1) + \bar{u}(2)) P(y(2) | \bar{u}(2))$$

for all $\bar{u}(i) \in \text{GF}(q) \times \text{GF}(q), y(i) \in \mathcal{Y}, i = 1, 2$.

In other words, this step can be viewed as defining two new 2-user q-ary MACs with extended output alphabets as follows:

$$\bar{U}(1) \xrightarrow{P^-} (Y(1), Y(2)), \quad \bar{U}(2) \xrightarrow{P^+} (Y(1), Y(2), \bar{U}(1))$$

With this construction, we observe that, analogous to the single-user channel polarization, the MAC channel also polarizes, and we have

$$I_J(P^-) \le I_J(P) \le I_J(P^+), \quad \forall J \subseteq E_2$$

We define a random process P_n, where $P_0 = P$ and $P_n = P_{n-1}^{B_n}$ for $n \ge 1$ with B_n distributed as i.i.d. uniform over $\{-, +\}$. We now state the results on polarization to extremal channels for this MAC.

Lemma 4: ([23] and [24])

1) The processes $\{I_J(P_n), J \subseteq E_2\}$ converges a.s., i.e., for each $J \subseteq E_2$, $I_J(\infty) \triangleq \lim_{n \to \infty} I_J(P_n)$ exists a.s.

2) With probability 1, $I_J(\infty) - I_{J-\{j\}}(\infty) \in \{0, 1\}, \forall J \subseteq E_2, j \in J$, where $I_\emptyset(\infty) = 0$.

From the above result, one can observe that the resulting achievable communication regions can be expressed as matroids, which describe the polarized states of the MAC channel. In particular, for a particular polarized state of the channel, the extremal channel state for the users in the MAC specifies which users' transmission should consist of information bits, and which users' bits are to be frozen. The coding scheme proposed in this study takes advantage of these extremal channels in aligning and decoding of the interference.

B. The One-Sided Interference Channel Case

We now restrict ourselves to the three user one-sided degraded interference channels, study the use of polar codes in this context, and show that this method can achieve the symmetric sum rate for such channels.

In order to use polar codes to communicate through these channels, we use point-to-point polar codes for each channel. For the channel from X_j to Y_j for $j = 1, 2$, described respectively as P_j, polarization of the channel for stages $i = 1, 2, \ldots$ is described with the following shorthand

$$I_j(i) \triangleq I[\{j\}](i), j = 1, 2. \tag{14}$$

Performing the polarization operation as described above yields polarized channels, which are either noiseless or completely noisy. Owing to the degraded channel assumption

	$q_{(0,0)}$	$q_{(1,0)}$	$q_{(1,1)}$
$I_2(i)$	0	1	1
$I_3(i)$	0	0	1

TABLE I
FRACTIONS OF POLARIZED EXTREMAL STATES FOR CHANNELS 2 AND 3
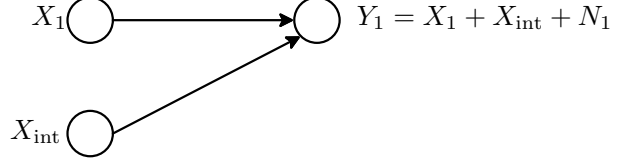

Fig. 2. The effective MAC channel at the first receiver

on P_2 and P_3, we observe that the these channels polarize simultaneously to yield possible $(I_2(i), I_3(i))$ tuples as $(0, 0)$, $(1, 0)$, and $(1, 1)$. We now define the fraction of channels polarized in each of these states as $q_{(0,0)}$, $q_{(1,0)}$ and $q_{(1,1)}$, respectively.

We can visualize the polarization of these channels from Table I. In particular, the 0 state corresponds to a situation where the symbol is frozen, while 1 signifies a noiseless channel where the information is sent. Since the fraction of polarized channels in P_j is $I(X_j; Y_j), j = 1, 2$, we have that

$$
\begin{aligned}
q_{(0,0)} + q_{(1,0)} + q_{(1,1)} &= 1 \\
q_{(1,0)} + q_{(1,1)} &= I(X_2; Y_2) \\
q_{(1,1)} &= I(X_3; Y_3)
\end{aligned}
$$

From the above equations, we can solve for $q_{(0,0)}$, $q_{(1,0)}$ and $q_{(1,1)}$ to obtain

$$
\begin{aligned}
q_{(1,1)} &= I(X_3; Y_3) \\
q_{(1,0)} &= I(X_2; Y_2) - I(X_3; Y_3) \\
q_{(0,0)} &= 1 - I(X_2; Y_2)
\end{aligned}
$$

In order to decode the combined interference at receiver 1, we represent the interference $aX_2 + aX_3$ as a single entity $X_{\text{int}} \in \text{GF}(q)$. In other words, we use the MAC formulation discussed above along with the following:

$$
\begin{aligned}
X_{\text{int}} &\triangleq aX_2 + aX_3 = a(GU_2 + GU_3) \\
&= G(U_{\text{int}}),
\end{aligned}
$$

where $U_{\text{int}} = a(U_2 + U_3)$.

Thus, the interference can be considered to come from a q-ary symbol transmitted in the effective MAC which is encoded with the same polar coding scheme as the one-sided interference channel.

We can now consider the effective channel to receiver 1 as an effective MAC channel as shown in Figure 2.

Now, using the results given in the previous subsection, we can conclude that the channel polarizes to extremal channels

which form a matroid. Our coding scheme at the polarized time instant i can be succinctly described as follows:

- If the channel $U_3(i) \rightarrow Y_3(i)$, defined as $P_3(i)$ is polarized to a 1 state, one can send information on the channel P_3. Due to the degraded assumption, the channel $P_2(i) : U_2(i) \rightarrow Y_2(i)$ can also transmit an information bit.
- If the channel $P_3(i)$ polarizes to a 0 state, we freeze $U_3(i)$ for this transmission. If P_2 polarized to a 1 state, an information bit is can be sent by $U_2(i)$. Otherwise, we set $U_2(i)$ as frozen as well.
- The transmission of information on $P_1 : X_1(i) \rightarrow Y_1(i)$ is governed by the effective MAC with $X_1(i), X_{\text{int}}(i) \rightarrow Y_1(i)$ described above. In this case, the transmission for $U_1(i)$ is identical to that of a user in a polar coded 2-user MAC.

The above coding strategy can be summarized by observing that channels P_2 and P_3 essentially utilize the point-to-point Polar coding strategy, while P_1 utilizes the 2-user MAC strategy with X_1 and X_{int} being the transmitting users. Based on this method, the achievable rate region can be given by the following.

Theorem 5: The achievable rates with polar coding on the degraded one-sided interference channel is given by the union of the non-negative rate tuples satisfying

$$
\begin{aligned}
R_1 &\leq I(X_1; Y_1, X_{\text{int}}) \\
R_2 &\leq I(X_2; Y_2) \\
R_3 &\leq I(X_3; Y_3) \\
R_1 + \max\{R_2, R_3\} &\leq I(X_1, X_{\text{int}}; Y_1)
\end{aligned}
$$

over uniformly distributed inputs.

Note that the additional constraint of $R_{\text{int}} \triangleq \max\{R_2, R_3\} \leq I(X_{\text{int}}; Y_1|X_1)$ for the MAC region is already satisfied due to the degradedness of the channel as $I(X_{\text{int}}; Y_1|X_1) \geq (X_2; Y_2) \geq I(X_3; Y_3)$.

We now provide an outer bound to the achievable rates in this setting.

Lemma 6: Any achievable rate tuple for the one-sided degraded interference channel belong to the region given by the union of non-negative rate tuples satisfying

$$
\begin{aligned}
R_1 &\leq I(X_1; Y_1|X_{\text{int}}) = I(X_1; Y_1, X_{\text{int}}) \\
R_2 &\leq I(X_2; Y_2) \\
R_3 &\leq I(X_3; Y_3) \\
R_1 + \max\{R_2, R_3\} &\leq I(X_1, X_{\text{int}}; Y_1),
\end{aligned}
$$

where each term on the right hand side is evaluated with uniform input probability distributions.

This can be observed by noting the following:

- Channels P_2 and P_3 are point-to-point links, on which the achievable rate is bounded by the mutual information. Thus, $R_j \leq I(X_j; Y_j), j = 1, 2$.
- The rate for channel P_1 is upper bounded by the rate obtainable if X_{int} is supplied to the receiver. In addi-

tion, since X_{int} and X_1 are independent, we get $R_1 \leq I(X_1; Y_1, X_{\text{int}})$.

- Due to the degraded assumption, the decodability of X_2 and X_3 from Y_2 and Y_3 respectively ensures the decodability of X_{int} at the first receiver. Thus, combining the interferer and the first transmitter, we obtain the bound $R_1 + \max\{R_2, R_3\} \leq I(X_1, X_{\text{int}}; Y_1)$. (We briefly note a more formal approach to this bound. Due to Fano's inequality one can bound $n(R_1 + R_2) \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n\epsilon$ with some $\epsilon \rightarrow 0$ as $n \rightarrow \infty$. Then, using degradedness and the fact that conditioning does not increase entropy we can obtain $I(X_2^n; Y_2^n) \leq I(X_{\text{int}}^n; Y_1^n|X_1^n)$, and use it to show $R_1 + R_2 \leq I(X_1, X_{\text{int}}; Y_1|Q)$ with a time sharing random variable Q.)
- Then, the outer bound region with the above inequalities can be stated with time sharing random variable Q, and a union over distributions $p(q)p(x_1|q)p(x_2|q)p(x_3|q)$. However, without loss of generality we can consider uniform inputs (independent of Q) as right hand side expressions at each of the inequalities is maximized by such a choice.

Corollary 7: From Lemma 5 and Lemma 6, we can conclude that the sum capacity of the channel is

$$
C_{\text{sum}} = I(X_1, X_{\text{int}}; Y_1) + \min\{I(X_2, Y_2), I(X_3, Y_3)\},
$$

with uniform input distributions $p(x_1)p(x_2)p(x_3)$.

V. DISCUSSION

Note the following facts:

- The scheme described above is optimal in terms of the sum capacity for the given channel. This way, it extends the Han-Kobayashi region, and there exist cases for which it extends the interference-decoding (with random codes) region described in [38]. Our region clearly includes the interference decoding region given in Corollary 2 when the inputs are uniform, as the region given for the latter has an additional sum rate constraint. In particular, the last inequality in the region described in Corollary 2 simplifies to $R_{\text{sum}} = R_1 + R_2 + R_3 \leq I(X_1, aX_2 + aX_3; Y_1)$ when $R_2 + R_3 \leq H(aX_2 + aX_3))$, $R_2 \leq H(aX_2)$, and $R_3 \leq H(aX_3)$. Thus, for a given channel, one can have $R_{\text{sum}} < C_{\text{sum}}$. Note that, from the converse analysis, we see that the the maximum values for each expression is achieved by the uniform distribution of inputs $p(x_1)$, $p(x_2)$, $p(x_3)$. Any deviation of $p(x_k)$ from the uniform distribution may degrade the achieved sum rate as it will decrease the corresponding R_k. This shows that the points achieved by our scheme strictly extends the two random coding regions given above.
- Note that the use of structured codes along with joint decoding of interference is by no means a new observation. For example, [35]–[37] show that lattices can provide a rate improvement over existing random coding strategies. The polar coding construction in this paper goes well beyond the conventional lattice/linear-coding

alignment strategies in [35]–[37]. Polar coding *induces* and extricates deterministic structure in noisy channels where structure may be not otherwise be easy to isolate. In addition, it does so using low complexity encoding and decoding. In particular, the encoding and decoding complexity of point-to-point and multiple access polar codes scale as $O(n \log n)$ [1], [8].

- The random coding rate regions in [] take into account all possible input distributions, including non-uniform. Therefore, it is of interest to extend the coding scheme to non-uniform input distributions for channel models different than the one considered in this paper. The polar coding method uses uniformly distributed information variables, $U_k(i)$ for user k at time index i. This results in uniformly distributed channel inputs X_k for user k. Because of this construction, the polar coding achieves the "symmetric capacity" of point-to-point channels [1], [8], and the "uniform rate region" of multiple access channels [23], [24]. However, using Gallager's method [39] one can easily construct arbitrary input distributions. For example, say $q = 2$ and hence $\mathcal{X}_1 = GF(2)$, and an input distribution with $\Pr\{X_1 = 0\} = \frac{1}{3}$ and $\Pr\{X_1 = 1\} = \frac{2}{3}$ is needed. Then, we can extend the channel input cardinality to $q' = 3$, and construct polar codes for the ternary input channel. Denoting the resulting codewords as X_1', we can map these uniformly distributed input distributions to X_1 with a mapping $f(\cdot) : GF(q') \rightarrow GF(q)$ with $f(0) = 0$ and $f(1) = f(2) = 1$. This way, the channel inputs $x_1(1) = f(x_1'(1)), \cdots, x_1(n) = f(x_1'(n))$ will have the desired distribution $p(x_1)$. Similarly, by increasing q' and using an appropriate function $f()$, one can obtain any desired $p(x_1)$ using Gallager's method. And, this way, it can be shown that polar coding can achieve the capacity of arbitrary input arbitrary output point-to-point DMCs [8]. However, it is not this straightforward to obtain non-uniform inputs in our coding scheme. In particular, considering the extended input alphabets, where $X_k, U_k \in GF(q')$, and the mappings f_k to construct desired input distributions with the Gallager's method, the interference signal is given by

$$X_{\text{int}} = a f_2(GU_2) + a f_3(GU_3), \qquad (15)$$

where the operations of GU_k are in $GF(q')$, and the summation and multiplications (with a) are in $GF(q)$. Due to this, it may not be possible to write $X_{\text{int}} = GU_{\text{int}}$ for the given functions f_k. This is a challenge, as the combined interference may not be treated as one polar code, and is an important direction of further research on this coding scheme. The main question here is: Can we find mappings $f_k(\cdot)$ for some q', that not only result in the desired input probability distributions $p_k()$, but also allow the interfering signals to be treated as belonging to one polar code?

- The polar coding approach discussed in this paper is not a specific scheme that pertains only to the considered channel model. In fact, we believe that polar coding is a

key to implement interference alignment and interference decoding in networks. To further elaborate on this, lets consider a MAC seen by receiver 1 ($\mathcal{P}_1$), which has an output Y_1. When we analyze the extremals of this channel (i.e., $I[J](\mathcal{P}_1)$ for each subset of users J), we observe that opportunities for interference alignment and decoding arises naturally. Remarkably, using the results of [24], we can view each extremal as a deterministic channel, where the channel output is given by $Y_1(i) = A_1(i)[X_1(i)X_2(i)X_3(i)]^T$ with $A_1(i)$ describing the deterministic channel at time index i. For example, one can have the following cases at two instants of the polarized channels:

$$A_1(j) = [100; 011], A_1(k) = [111; 011], \qquad (16)$$

with the corresponding outputs are given by

$$
\begin{aligned}
Y_1(j) &= [X_1(j); X_2(j) + X_3(j)], \\
Y_1(k) &= [X_1(k) + X_2(k) + X_3(k); X_2(k) + X_3(k)].
\end{aligned}
$$

Clearly, the time index j corresponds to an interference alignment time index, and, one can implement both interference alignment and decoding at time index k. This suggests that the coding over interference networks can utilize polar coding over MACs, which can be used for alignment and decoding of the interference. And, to determine the achievable region over such a construction, one needs to find the frequencies of the extremals for a given channel. This is an ongoing research that we are pursuing at the moment.

- Another point that we want to include that whether one might be interested in designing different polar coding matrices (i.e., G_ks) in the interference channel context compared to point-to-point and multiple-access scenarios. Can one design G_ks specific to channels given by $p(y_k|x_1, x_2, x_3)$ that aligns the interfering signals and allow interference decoding?

VI. CONCLUSION

We introduce a polar coding based alignment scheme for interference networks, with a focused application to one-side degraded interference channels. This enables deterministic (and in our examples, linear) structure on the resulting channels, over which we perform alignment by decoding the *sum-interference*. Such a coding scheme is found to outperform existing random coding based approaches for this channel. Additional side-benefits of the polar coding based scheme include low-complexity encoding and decoding, and deterministic (constructive) encoding and decoding arguments. We believe that considerable work needs to be done to fully uncover the potential gains of bringing polar coding and alignment together in one framework.

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