A New Achievable Rate Region for the Discrete Memoryless X Channel

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***Abstract*—We consider the discrete memoryless X channel, a communication model with two transmitters and two receivers in which every transmitter has a message for every receiver. We propose an achievable scheme, based on the message splitting and binning techniques, which results in the best inner bound on the capacity region of the X channel to date.**

I. Introduction

In this paper we propose a signaling scheme for two-user X channels. The X channel refers to a communication scenario in which each transmitter has a message for every receiver. This model involves most of the multi-user channels studied in information theory; such as multiple access channel, broadcast channel and interference channel.

Recently the degrees of freedom region for the MIMO X channel is characterized in [1]–[3]. It is established that for the MIMO X channel with $M > 1$ number of antennas at all nodes and with non-degenerate channel matrices, the degrees of freedom is equal to $\frac{4}{3}M$. The MIMO X channel is the first known example that has non-integer degrees of freedom and has received much attention lately. In this paper, we propose a signaling scheme for the X channel using message splitting and binning. The proposed scheme uses the message splitting technique to split messages in two parts (common and private) and utilizes the common messages in the construction of the cloud centers ([4]–[6]), which are used to design superposition codes. A binning technique is used for the private messages at the transmitters to allow coding in the sense of [7] (see also [8]). In the special case of broadcast channel, our proposed rate region reduces to the best known achievable region for the two user broadcast channel [4], in the case of interference channel it reduces to that of [9] (a simpler description of this region is recently given in [10], see also [11]) and in the case of multiple access channel it achieves the capacity region (see, e.g,. [12]).

The proposed region outperforms that of [2], which to the best of our knowledge is the best proposed achievable region for the X channel to date. For example, in the special case of degraded broadcast channel the proposed scheme is capacity achieving whereas [2] is not. In addition, in the case of interference channel [2] reduces to the scheme of considering interference as noise, whereas the proposed region allows for interference cancellation and achieves the Han and Kobahayashi region ([9]).

The remaining sections of the paper are organized as follows. In Section II, we provide the notation used throughout the sequel, and in Section III we describe the system model. Section IV is devoted to the main result of the paper and some concluding remarks are provided in Section V. Proofs are collected in the Appendix to enhance the flow of the paper.

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II. Notation

The notations used in the rest of the paper is described in this section. Vectors are denoted as $\mathbf{x}^i = \{x(1), \cdots, x(i)\}$, where we omit the i if $i = n$, i.e., $\mathbf{x} = \{x(1), \cdots, x(n)\}$. Random variables are denoted with capital letters (X), and random vectors are denoted by bold-capital letters ($\mathbf{X}^i$). Again, we drop the i for $\mathbf{X} = \{X(1), \cdots, X(n)\}$. Finally, $P(X = x)$ is denoted by $p(x)$, where we omit the random variable X.

III. System Model

We consider a two-user discrete memoryless X channel (XC), comprised of two transmitter-receiver pairs (see Fig. 1), and is denoted by

$$(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_2),$$

for some finite sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$. Here the symbols $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ are the channel inputs and the symbols $(y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2$ are the channel outputs observed at the decoder 1 and decoder 2, respectively. The channel is memoryless and time-invariant:

$$p(y_1(t), y_2(t)|\mathbf{x}_1^t, \mathbf{x}_2^t, \mathbf{y}_1^{t-1}, \mathbf{y}_2^{t-1}) = p(y_1(t), y_2(t)|x_1(t), x_2(t)).$$

We assume that each transmitter $k \in \{1, 2\}$ has messages W_{k1} and W_{k2} which is to be transmitted to receiver 1 and receiver 2, respectively, in n channel uses. In this setting, we define $(n, M_{11}, M_{12}, M_{21}, M_{22}, P_{e,1}^{(n)}, P_{e,2}^{(n)})$ codebook with the following components:

1) The message sets $\mathcal{W}_{k1} = \{1, ..., M_{k1}\}$ and $\mathcal{W}_{k2} = \{1, ..., M_{k2}\}$ for transmitter $k = 1, 2$.

2) An encoding function $f_k(.)$ at transmitter k which maps the messages to the transmitted symbols, $f_k : (w_{k1}, w_{k2}) \rightarrow \mathbf{X}_k$ for each $(w_{k1}, w_{k2}) \in \mathcal{W}_{k1} \times \mathcal{W}_{k2}$ for $k = 1, 2$.

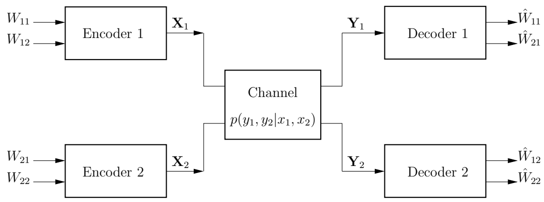

Fig. 1. The two user discrete memoryless X channel.

3) Decoding function $\phi_k(.)$ at receiver k which maps the received symbols to an estimate of the message: $\phi_k(\mathbf{Y}_k) = (\hat{w}_{1k}, \hat{w}_{2k})$ for $k = 1, 2$.

4) Reliability of the transmission for receiver k is measured by $P_{e,k}^{(n)}$, where

$$P_{e,k}^{(n)} = \frac{1}{M_{1k}M_{2k}} \sum_{(w_{1k}, w_{2k}) \in \mathcal{W}_{1k} \times \mathcal{W}_{2k}} Pr\{\phi_k(\mathbf{Y}_k) \neq (w_{1k}, w_{2k}) \\ |(w_{11}, w_{12}, w_{21}, w_{22}) \text{ is sent}\},$$

for $k = 1, 2$.

We say that the rate tuple $(R_{11}, R_{12}, R_{21}, R_{22})$ is achievable for the X channel, if, for any given $\epsilon > 0$, there exists an $(n, M_{11}, M_{12}, M_{21}, M_{22}, P_{e,1}^{(n)}, P_{e,2}^{(n)})$ codebook such that,

$$\begin{aligned}
\frac{1}{n}\log(M_{11}) &= R_{11}, \\
\frac{1}{n}\log(M_{12}) &= R_{12}, \\
\frac{1}{n}\log(M_{21}) &= R_{21}, \\
\frac{1}{n}\log(M_{22}) &= R_{22}, \\
\max\{P_{e,1}^{(n)}, P_{e,2}^{(n)}\} &\leq \epsilon,
\end{aligned}$$

for sufficiently large n. The capacity region is the closure of the set of all achievable rate pairs (R_1, R_2) and is denoted by $\mathbb{C}^{\text{XC}}$.

IV. Main Result

The proposed scheme is based on message splitting and binning. First we split the message of transmitter i to receiver j, W_{ij}, into the following: 1) A common message W_{ijc}, and 2) A private message W_{ijp}. We require receivers to decode all common messages of the transmitters and the intended private messages.

The encoding procedure is explained in the following: The common messages are used for superposition coding, the codeword of which serves as "cloud centers" [5] (see also [6], [12]) for the remaining random variables. However, a general encoding approach is considered (see, e.g., [4]). The common message allows for partial interference cancellation in the sense of [9] as we require joint decoding at the transmitters. Finally, we use the binning technique of [8] to jointly encode the private messages. This allows to design the private messages, part of which can be considered as non-causally known interference in the sense of [7]. The main result of the paper is given below.

Theorem 1: Let $\mathcal{P}$ be the set of probability distributions $p(.)$ that factor as

$$\begin{aligned}
&p(q, v_{1c}, v_{11p}, v_{12p}, v_{2c}, v_{21p}, v_{22p}, x_1, x_2) \\
&= p(q)p(v_{1c}, v_{11p}, v_{12p}|q)p(v_{2c}, v_{21p}, v_{22p}|q) \\
&\quad p(x_1|v_{1c}, v_{11p}, v_{12p}, q)p(x_2|v_{2c}, v_{21p}, v_{22p}, q). \qquad (1)
\end{aligned}$$

For any $p \in \mathcal{P}$, $\mathcal{R}_I(p)$ is the set of non-negative rate tuples $(R_{11c}, R_{11p}, R_{12c}, R_{12p}, R_{21c}, R_{21p}, R_{22c}, R_{22p})$ satisfying

$$\begin{aligned}
R_{1c} + R_{11p^*} &< I(V_{1c}, V_{11p}; Y_1|V_{2c}, V_{21p}, Q) \\
R_{11p^*} &< I(V_{11p}; Y_1|V_{1c}, V_{2c}, V_{21p}, Q) \\
R_{21p^*} &< I(V_{21p}; Y_1|V_{1c}, V_{2c}, V_{11p}, Q) \\
R_{2c} + R_{21p^*} &< I(V_{2c}, V_{21p}; Y_1|V_{1c}, V_{11p}, Q) \\
R_{1c} + R_{11p^*} + R_{21p^*} &< I(V_{1c}, V_{11p}, V_{21p}; Y_1|V_{2c}, Q) \\
R_{11p^*} + R_{21p^*} &< I(V_{11p}, V_{21p}; Y_1|V_{1c}, V_{2c}, Q) \\
R_{1c} + R_{2c} + R_{11p^*} + R_{21p^*} &< I(V_{1c}, V_{2c}, V_{11p}, V_{21p}; Y_1|Q) \\
R_{2c} + R_{11p^*} + R_{21p^*} &< I(V_{2c}, V_{11p}, V_{21p}; Y_1|V_{1c}, Q) \\
R_{11p} + R_{12p} &< R_{11p^*} + R_{12p^*} \\
&\quad - I(V_{11p}; V_{12p}|V_{1c}, Q) \\
R_{21p} + R_{22p} &< R_{21p^*} + R_{22p^*} \\
&\quad - I(V_{21p}; V_{22p}|V_{2c}, Q) \\
R_{1c} &= R_{11c} + R_{12c} \\
R_{2c} &= R_{21c} + R_{22c},
\end{aligned}$$

and

$$\begin{aligned}
R_{11p} \leq R_{11p^*} \quad &, \quad R_{12p} \leq R_{12p^*}, \\
R_{21p} \leq R_{21p^*} \quad &, \quad R_{22p} \leq R_{22p^*}. \qquad (2)
\end{aligned}$$

Similarly we define $\mathcal{R}_{II}(p)$, which is the set of non-negative tuples $(R_{11c}, R_{11p}, R_{12c}, R_{12p}, R_{21c}, R_{21p}, R_{22c}, R_{22p})$ satisfying equations (2) with the indices swapped everywhere.

For a set $\mathcal{S}$ of tuples $(R_{11c}, R_{11p}, R_{12c}, R_{12p}, R_{21c}, R_{21p}, R_{22c}, R_{22p})$, we define $\prod(\mathcal{S})$ as the set of tuples $(R_{11}, R_{12}, R_{21}, R_{22})$ such that $R_{11} = R_{11c} + R_{11p}$, $R_{12} = R_{12c} + R_{12p}$, $R_{21} = R_{21c} + R_{21p}$, and $R_{22} = R_{22c} + R_{22p}$.

The set

$$\mathcal{R} = \prod \left(\bigcup_{p \in \mathcal{P}} \mathcal{R}_I(p) \cap \mathcal{R}_{II}(p) \right) \qquad (3)$$

is an achievable region for the discrete memoryless XC.

Proof: Please refer to Appendix A. ■

Below we discuss the special cases of the rate region given above.

A. The Broadcast Channel

For a given BC $p(y_1, y_2|x_1)$, we consider the X channel given by $p(y_1, y_2|x_1, x_2) = p(y_1, y_2|x_1)$. For this special case, the region $\mathcal{R}_I(p) \cap \mathcal{R}_{II}(p)$ reduces to the following

$$\begin{aligned} R_{1c} + R_{11p^*} &< I(V_{1c}, V_{11p}; Y_1|Q) && (4)\\ R_{11p^*} &< I(V_{11p}; Y_1|V_{1c}, Q) && (5)\\ R_{1c} + R_{12p^*} &< I(V_{1c}, V_{12p}; Y_2|Q) && (6)\\ R_{12p^*} &< I(V_{12p}; Y_2|V_{1c}, Q) && (7)\\ R_{11p} + R_{12p} &< R_{11p^*} + R_{12p^*} \\ &\quad - I(V_{11p}; V_{12p}|V_{1c}, Q) && (8)\\ R_{1c} &= R_{11c} + R_{12c} && (9)\\ R_{11p} &\leq R_{11p^*} && (10)\\ R_{12p} &\leq R_{12p^*} && (11) \end{aligned}$$

for a given $p \in \mathcal{P}$, where we set V_{2c}, V_{21p}, V_{22p} to be deterministic (as channel input of transmitter 2 does not affect the received signals, this does not reduce the achievable rate region). We further set Q to be deterministic. Let $R_{11} = R_{11c} + R_{11p}$, $R_{12} = R_{12c} + R_{12p}$, by applying the Fourier-Motzkin elimination we obtain the following.

Any non-negative rate pair (R_{11}, R_{12}) satisfying

$$\begin{aligned} R_{11} &< I(V_{1c}, V_{11p}; Y_1) && (12)\\ R_{12} &< I(V_{1c}, V_{12p}; Y_2) && (13)\\ R_{11} + R_{12} &< \min\{I(V_{1c}; Y_1), I(V_{1c}; Y_2)\} \\ &\quad + I(V_{11p}; Y_1|V_{1c}) + I(V_{12p}; Y_2|V_{2c}) \\ &\quad - I(V_{11p}, V_{12p}|V_{1c}) && (14) \end{aligned}$$

for some $p(v_{1c}, v_{11p}, v_{12p})p(x_1|v_{1c}, v_{11p}, v_{12p})$ is achievable for the broadcast channel given by $p(y_1, y_2|x_1)$. This is the Marton's rate region [4] and is the best known inner bound to the capacity region of the two-user broadcast channels.

B. The Interference Channel

For the interference channel, the cross messages do not exist and hence we set random variables V_{12p}, V_{21p} to be deterministic. We also choose the common and private auxiliary random variables to be independent in the region $\mathcal{R}_I(p) \cap \mathcal{R}_{II}(p)$. Next, we set $R_{11} = R_{11c} + R_{11p}$, $R_{22} = R_{22c} + R_{22p}$, $R_{12c} = R_{12p} = R_{21c} = R_{21p} = 0$, and apply Fourier-Motzkin elimination to the obtained region and choose $p(x_1|v_{1c}, v_{11p}, q)$, $p(x_2|v_{2c}, v_{22p}, q)$ to be deterministic functions. We obtain the following region.

Any non-negative rate pair (R_{11}, R_{22}) satisfying

$$\begin{aligned} R_{11} &< I(X_1; Y_1|V_{2c}, Q)\\ R_{11} &< I(X_1; Y_1|V_{1c}, V_{2c}, Q) + I(V_{1c}; Y_2|X_2, Q)\\ R_{22} &< I(X_2; Y_2|V_{1c}, Q)\\ R_{22} &< I(X_2; Y_2|V_{1c}, V_{2c}, Q) + I(V_{2c}; Y_1|X_1, Q)\\ R_{11} + R_{22} &< I(X_2; Y_2|V_{1c}, V_{2c}, Q) + I(X_1, V_{2c}; Y_1|Q)\\ R_{11} + R_{22} &< I(X_1; Y_1|V_{1c}, V_{2c}, Q) + I(X_2, V_{1c}; Y_2|Q)\\ R_{11} + R_{22} &< I(X_1, V_{2c}; Y_1|V_{1c}, Q)\\ &\quad + I(X_2, V_{1c}; Y_2|V_{2c}, Q)\\ 2R_{11} + R_{22} &< I(X_1; Y_1|V_{1c}, V_{2c}, Q) + I(X_1, V_{2c}; Y_1|Q)\\ &\quad + I(X_2, V_{1c}; Y_2|V_{2c}, Q)\\ R_{11} + 2R_{22} &< I(X_2; Y_2|V_{1c}, V_{2c}, Q) + I(X_2, V_{1c}; Y_2|Q)\\ &\quad + I(X_1, V_{2c}; Y_1|V_{1c}, Q) \end{aligned}$$

for some $p(q)p(v_{1c}|q)p(v_{11p}|q)p(x_1|v_{1c}, v_{11p}, q)p(v_{2c}|q)$ $p(v_{22p}|q)p(x_2|v_{2c}, v_{22p}, q)$, is achievable for the interference channel given by $p(y_1, y_2|x_1, x_2)$.

This region is the region given in Lemma 1 of [11], which can be shown to be equal to the compact form of Han and Kobahayashi rate region (see also [10]) by utilizing Lemma 2 of [11].

We remark that the scheme of [11] does not consider the event of not correctly decoding unintended common messages at the receivers as an error event. However, in deriving our rate region for the X channel, we consider that event as a decoding error. Interestingly, applying the Fourier-Motzkin elimination to the proposed rate region, for the special case of interference channel, results in the compact form of the Han and Kobayashi rate region. Therefore, the exclusion of the aforementioned error event does not affect the result obtained in [11].

V. Conclusion

A new achievable rate region for the two-user X channel is established. The proposed scheme is based on a combination of the binning technique for broadcast channels (see, e.g., [8], [6]) and the message splitting for interference channels considered in [9] with joint decoding. The proposed method generalizes the one proposed in [2] and, to the best of our knowledge, achieves the largest region for the two user discrete memoryless X channel. As a future work, simplification of the achievable rate region, finding tight outer bounds, and studying networks involving more than two-users are of definite interest.

Appendix A
Proof of Achievability

First we fix $p(q)$, $p(v_{1c}, v_{11p}, v_{12p}|q)$, $p(v_{2c}, v_{21p}, v_{22p}|q)$, $p(x_1|v_{1c}, v_{11p}, v_{12p}, q)$, $p(x_2|v_{2c}, v_{21p}, v_{22p}, q)$, and the channel is given by $p(y_1, y_2|x_1, x_2)$.

We then generate a random typical sequence $\mathbf{q}$, where $p(\mathbf{q}) = \prod_{i=1}^{n} p(q^{(i)})$ and each entry is chosen i.i.d. according to $p(q)$. Every node knows the sequence $\mathbf{q}^n$. Below we describe the codebook generation and encoding for transmitter 1. We follow a similar procedure at transmitter 2. Please refer to Fig. 2 for a depiction of the encoder structure.

Codebook Generation:

Each codebook in the ensemble is constructed as follows. We first split the message W_{11}, which is to be decoded at the receiver 1, as $W_{11} = \{W_{11c}, W_{11p}\}$, where W_{11c} and W_{11p} are the common and the private messages of transmitter 1 destined to receiver 1, and split message W_{12}, which is to be decoded at the receiver 2, as $W_{12} = \{W_{12c}, W_{12p}\}$, where

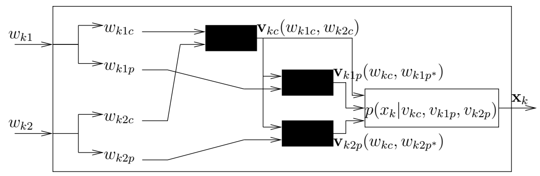

Fig. 2. The proposed encoder structure for transmitter k. Black boxes denote the proposed codebooks.

W_{12c} and W_{12p} are the common and the private messages of transmitter 1 intended for receiver 2.

$$W_{11c} = [1, 2, \cdots, 2^{nR_{11c}}] \tag{15}$$

$$W_{11p} = [1, 2, \cdots, 2^{nR_{11p}}] \tag{16}$$

$$W_{12c} = [1, 2, \cdots, 2^{nR_{12c}}] \tag{17}$$

$$W_{12p} = [1, 2, \cdots, 2^{nR_{12p}}] \tag{18}$$

Similarly we have W_{21c}, W_{21p}, W_{22c}, and W_{22p} for transmitter 2.

We generate $2^{nR_{1c}}$ i.i.d. sequences $\mathbf{v}_{1c}(w_{11c}, w_{12c})$, where

$$R_{1c} = R_{11c} + R_{12c}, \tag{19}$$

according to the distribution $\prod_{t=1}^{n} p(v_{1c}(t)|q(t))$, where the tuple (w_{11c}, w_{12c}) gives the codeword index denoted by $w_{1c} \in \{1, \cdots, 2^{n(R_{11c}+R_{12c})}\}$. In the sequel, we also denote these codewords with $\mathbf{v}_{1c}(w_{1c})$.

For each $\mathbf{v}_1(w_{1c})$, we generate $2^{nR_{11p^*}}$ i.i.d. sequences $\mathbf{v}_{11p}(w_{1c}, w_{11p}, w_{11p'})$ according to the distribution $\prod_{t=1}^{n} p(v_{11p}(t)|v_{1c}(t), q(t))$ and randomly throw them into $2^{nR_{11p}}$ bins, where we choose

$$R_{11p} \leq R_{11p^*} \tag{20}$$

Here, $w_{11p} \in \{1, 2, \cdots, 2^{nR_{11p}}\}$ denotes the bin index and $w_{11p'}$ denotes the codeword index within a particular bin. Combining these two we also enumerate the codewords with $\mathbf{v}_{11p}(w_{1c}, w_{11p}^*)$, where $w_{11p}^* \in \{1, 2, \cdots, 2^{nR_{11p^*}}\}$.

Similarly, we generate $2^{nR_{12p^*}}$ i.i.d. sequences $\mathbf{v}_{12p}(w_{1c}, w_{12p}, w_{12p'})$ according to the distribution $\prod_{t=1}^{n} p(v_{12p}(t)|v_{1c}(t), q(t))$, and randomly throw them into $2^{nR_{12p}}$ bins, where we choose

$$R_{12p} \leq R_{12p^*} \tag{21}$$

Here, $w_{12p} \in \{1, 2, \cdots, 2^{nR_{12p}}\}$ denotes the bin index and $w_{12p'}$ denotes the codeword index of a particular bin. Combining these two we also enumerate the codewords with $\mathbf{v}_{12p}(w_{1c}, w_{12p}^*)$, where $w_{12p}^* \in \{1, 2, \cdots, 2^{nR_{12p^*}}\}$.

Second transmitter uses a similar strategy to generate the following sequences: $\mathbf{v}_{2c}(w_{21c}, w_{22c})$, $\mathbf{v}_{21p}(w_{2c}, w_{21p}, w_{21p'})$, and $\mathbf{v}_{22p}(w_{2c}, w_{22p}, w_{22p'})$, where we require

$$\begin{aligned} R_{21p} &\leq R_{21p^*} \\ R_{22p} &\leq R_{22p^*} \\ R_{2c} &= R_{21c} + R_{22c}. \end{aligned} \tag{22}$$

Encoding:

To transmit message tuple (w_{11}, w_{12}), transmitter 1 first splits them into $(w_{11c}, w_{11p}, w_{12c}, w_{12p})$. Then, it looks for codewords $\mathbf{v}_{11p}$ in the bin w_{11p} and codewords $\mathbf{v}_{12p}$ in the bin w_{12p}, respectively, satisfying

$$\begin{aligned} (\mathbf{q}, \mathbf{v}_{1c}(w_{1c}), \mathbf{v}_{11p}(w_{1c}, w_{11p}, j), \mathbf{v}_{12p}(w_{1c}, w_{12p}, k)) \\ \in \mathcal{A}_\epsilon^{(n)}(Q, V_{1c}, V_{11p}, V_{12p}). \end{aligned} \tag{23}$$

Here indices j and k denote codeword indices within the given bins. If there is no such pair of codewords, then an encoding error will be declared. If there is more than one pair, then one is randomly chosen. After finding such a tuple, the encoder generates the channel input $\mathbf{x}_1$ according to $p(\mathbf{x}_1) = \prod_{t=1}^{n} p(x_1(t)|v_{1c}(t), v_{11p}(t), v_{12p}(t), q(t))$.

Similarly, transmitter 2 generates its channel input $\mathbf{x}_2$.

Decoding:

In the following we describe the decoding strategy for reciever 1. Similar steps are taken at receiver 2. Reciever 1 tries to obtain the estimates $(\hat{w}_{11c}, \hat{w}_{11p}, \hat{w}_{21c}, \hat{w}_{21p})$ to construct the message estimates $(\hat{w}_{11}, \hat{w}_{21})$. Accordingly, it looks looks for tuples $(w_{1c}, w_{11p^*}, w_{2c}, w_{21p^*})$ satisfying

$$\begin{aligned} (\mathbf{q}, \mathbf{v}_{1c}(w_{1c}), \mathbf{v}_{2c}(w_{2c}), \mathbf{v}_{11p}(w_{1c}, w_{11p^*}), \mathbf{v}_{21p}(w_{2c}, w_{21p^*}), \mathbf{y}_1) \\ \in \mathcal{A}_\epsilon^{(n)}(Q, V_{1c}, V_{2c}, V_{11p}, V_{21p}, Y_1) \end{aligned} \tag{24}$$

If such tuple exists with unique indices, it will first obtain w_{11c} from w_{1c}, w_{21c} from w_{2c}, w_{11p} from w_{11p^*}, and w_{21p} from w_{21p^*}, then it will set $\hat{w}_{11c} = w_{11c}$, $\hat{w}_{21c} = w_{21c}$, $\hat{w}_{11p} = w_{11p}$ and $\hat{w}_{21p} = w_{21p}$; otherwise it will declare an error. After estimating $(\hat{w}_{11c}, \hat{w}_{21c}, \hat{w}_{11p}, \hat{w}_{21p})$ the receiver will obtain the corresponding message estimates $\hat{w}_{11}$ and $\hat{w}_{21}$.

Error Probability Analysis:

We first focus on error probability $P_{e,1}^{(n)}$. Without loss of generality and by the symmetrical property of the ensemble it suffices to consider $w_{11} = w_{12} = w_{21} = w_{22} = 1$ is transmitted. We also assume that, if there is no encoding error, the first codewords in the bins are chosen at the encoders (for example, $j = k = 1$ in (23)). We consider the following events.

E_1 : There is no pair $(\mathbf{v}_{11p}, \mathbf{v}_{12p})$ such that $(\mathbf{q}, \mathbf{v}_{1c}(1), \mathbf{v}_{11p}(1, 1, k_1), \mathbf{v}_{12p}(1, 1, j_1)) \in \mathcal{A}_\epsilon^{(n)}(Q, V_{1c}, V_{11p}, V_{12p})$

E_2 : There is no pair $(\mathbf{v}_{21p}, \mathbf{v}_{22p})$ such that $(\mathbf{q}, \mathbf{v}_{2c}(1), \mathbf{v}_{21p}(1, 1, j_2), \mathbf{v}_{22p}(1, 1, k_2)) \in \mathcal{A}_\epsilon^{(n)}(Q, V_{2c}, V_{21p}, V_{22p})$

E_3 : $(\mathbf{q}, \mathbf{v}_{1c}(1), \mathbf{v}_{2c}(1), \mathbf{v}_{11p}(1, 1, 1), \mathbf{v}_{21p}(1, 1, 1), \mathbf{y}_1)$ does not satisfy (24)

E_4 : $(\mathbf{q}, \mathbf{v}_{1c}(i_1), \mathbf{v}_{2c}(i_2), \mathbf{v}_{11p}(i_1, k_1^*), \mathbf{v}_{21p}(i_2, k_2^*), \mathbf{y}_1)$ satisfies (24) with $(i_1, i_2, k_1^*, k_2^*) \neq (1, 1, 1, 1)$

From the analysis of encoding error probability in [6], [8], $Pr\{E_1\} \leq \epsilon$ as $n \to \infty$, if

$$R_{11p} + R_{12p} < R_{11p^*} + R_{12p^*} - I(V_{11p}; V_{12p}|V_{1c}, Q). \tag{25}$$

Similarly $Pr\{E_2\} \leq \epsilon$ as $n \to \infty$, if

$$R_{21p} + R_{22p} < R_{21p^*} + R_{22p^*} - I(V_{21p}; V_{22p}|V_{2c}, Q). \quad (26)$$

Asymptotic equipartition property (see, e.g., [12]) assures that $Pr\{E_3\} \leq \epsilon$ for sufficiently large n.

It remains to show the conditions for which $Pr\{E_4|E_3^c\} \leq \epsilon$ for sufficiently large n, as $P_{e,1}^{(n)} \leq Pr\{E_1\} + Pr\{E_2\} + Pr\{E_3\} + Pr\{E_4|E_3^c\}$. We first define the following event

$$E_4(\mathbf{i}) = \big\{(\mathbf{q}, \mathbf{v}_{1c}(i_1), \mathbf{v}_{2c}(i_2), \mathbf{v}_{11p}(i_1, k_1^*), \mathbf{v}_{21p}(i_2, k_2^*), \mathbf{y}_1) \in \mathcal{A}_\epsilon^{(n)}(Q, V_{1c}, V_{2c}, V_{11p}, V_{21p}, Y_1)\big|E_3^c\big\}$$

where the index vector is given by $\mathbf{i} = \{i_1, i_2, k_1^*, k_2^*\}$. Then, using the Boole's inequality (a.k.a, the union bound), we write

$$\begin{aligned}
Pr\{E_4|E_3^c\} &= Pr\left\{\bigcup_{(i_1,i_2,k_1^*,k_2^*)\neq(1,1,1,1)} E_4(\mathbf{i})\right\} \\
&\leq \sum_{\substack{i_1\neq1, i_2=1 \\ k_1^*=1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2\neq1 \\ k_1^*=1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2\neq1 \\ k_1^*=1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2=1 \\ k_1^*\neq1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2=1 \\ k_1^*\neq1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2\neq1 \\ k_1^*\neq1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2\neq1 \\ k_1^*\neq1, k_2^*=1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2=1 \\ k_1^*=1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2=1 \\ k_1^*=1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2\neq1 \\ k_1^*=1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2\neq1 \\ k_1^*=1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2=1 \\ k_1^*\neq1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2=1 \\ k_1^*\neq1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} \\
+ \sum_{\substack{i_1\neq1, i_2\neq1 \\ k_1^*\neq1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\} &+ \sum_{\substack{i_1=1, i_2\neq1 \\ k_1^*\neq1, k_2^*\neq1}} Pr\{E_4(\mathbf{i})\}
\end{aligned}$$

From joint typicality results (see, e.g., [12]), we can show that $Pr\{E_4|E_3^c\}$ vanishes for sufficiently large n, once the rates satisfy the following equations.

$$\begin{aligned}
R_{1c} &< I(V_{1c}, V_{11p}; Y_1|V_{2c}, V_{21p}, Q) && (27) \\
R_{1c} + R_{2c} &< I(V_{1c}, V_{11p}, V_{2c}, V_{21p}; Y_1|Q) && (28) \\
R_{2c} &< I(V_{2c}, V_{21p}; Y_1|V_{1c}, V_{11p}, Q) && (29) \\
R_{1c} + R_{11p^*} &< I(V_{1c}, V_{11p}; Y_1|V_{2c}, V_{21p}, Q) && (30) \\
R_{11p^*} &< I(V_{11p}; Y_1|V_{1c}, V_{2c}, V_{21p}, Q) && (31)
\end{aligned}$$

$$\begin{aligned}
&R_{1c} + R_{11p^*} \\
&\qquad + R_{2c} < I(V_{1c}, V_{11p}, V_{2c}, V_{21p}; Y_1|Q) && (32) \\
&R_{11p^*} + R_{2c} < I(V_{11p}, V_{2c}, V_{21p}; Y_1|V_{1c}, Q) && (33) \\
&R_{1c} + R_{21p^*} < I(V_{1c}, V_{11p}, V_{21p}; Y_1|V_{2c}, Q) && (34) \\
&R_{21p^*} < I(V_{21p}; Y_1|V_{1c}, V_{2c}, V_{21p}, Q) && (35) \\
&R_{1c} + R_{2c} \\
&\qquad + R_{21p^*} < I(V_{1c}, V_{11p}, V_{2c}, V_{21p}; Y_1|Q) && (36) \\
&R_{2c} + R_{21p^*} < I(V_{2c}, V_{21p}; Y_1|V_{1c}, V_{11p}, Q) && (37) \\
&R_{1c} + R_{11p^*} \\
&\qquad + R_{21p^*} < I(V_{1c}, V_{11p}, V_{21p}; Y_1|V_{2c}, Q) && (38) \\
&R_{11p^*} + R_{21p^*} < I(V_{11p}, V_{21p}; Y_1|V_{1c}, V_{2c}, Q) && (39) \\
&R_{1c} + R_{2c} \\
&\qquad + R_{11p^*} + R_{21p^*} < I(V_{1c}, V_{2c}, V_{11p}, V_{21p}; Y_1|Q) && (40) \\
&R_{2c} + R_{11p^*} \\
&\qquad + R_{21p^*} < I(V_{2c}, V_{11p}, V_{21p}; Y_1|V_{1c}, Q) && (41)
\end{aligned}$$

Noting that (27), (28), (29), (32), (33), (34), and (36) are redundant, we obtain that $P_{e,1}^{(n)}$ vanishes as n increases if (19), (20), (21), (22), (25), (26), (30), (31), (35), (37), (38), (39), (40), (41) are satisfied, which gives the rate region defined by $\mathcal{R}_I(p)$.

Similarly $P_{e,2}^{(n)}$ vanishes for sufficiently large n, if the rates belong to the region $\mathcal{R}_{II}(p)$. Finally, it can be readily observed that, for a given p, any rate tuple inside the region $\mathcal{R}_I(p) \bigcap \mathcal{R}_{II}(p)$ is achievable, which concludes the proof of the theorem.

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