Joint Interference Cancellation and Dirty Paper Coding for Cognitive Cellular Networks

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***Abstract*—Downlink communication in a cellular network with a cognitive (secondary) cell is considered. In our model, the base station of the cognitive cell knows the messages of the other cell non-causally. We propose a new interference cancellation technique that zero forces the intra-cell interference in the primary cell by the help of the cognitive base station. In addition, as the primary messages are known at the cognitive base station, the interference caused by the primary base station on the secondary users are canceled using dirty paper coding (DPC). Moreover, we provide an outer bound on the achievable degrees of freedom (DoF) region and show that for some special cases the proposed signaling scheme is sum DoF optimal for the considered system when the cognitive cell operates at its maximum sum DoF. The benefit of the cognitive paradigm is also established using the derived outer-bound and showing that the achieved sum DoF is strictly larger than the case when cognitive message sharing is unavailable.**

I. Introduction

A significant factor in limiting the performance of cellular systems is the interference from other cells also known as *inter-cell interference*. Interference from other users also degrades the achievable throughput in a K-user MIMO interference channel. An important technique proposed to mitigate these effects is interference alignment (IA) [1], [2]. Roughly speaking, IA *aligns* all the unwanted (interfering) signals to certain dimensions allowing the intended messages to be communicated over the remaining interference free ones. To achieve the gains promised by IA, the users need to have perfect knowledge of each others channel state information (CSI). That is, all the transmitters need to know all the channel realizations before forming their signals. Because the CSI needs to be obtained through training sequences and feedback, this introduces a serious overhead to the system. In [3], [4] the authors have considered the DoF region of MIMO networks in the absence of CSI at the transmitters. They have established the negative result that in most cases the degrees of freedom region can be achieved by simple time sharing which means nothing can be gained beyond time division access.

In this paper, we consider the problem of interference management for a cellular system in the downlink when the CSI is not fully available at all of the transmitters. We also assume one of the base stations to be *cognitive*. The cognitive message sharing means that the messages of one cell's (henceforth the non-cognitive or primary cell) are made available non-causally to the other one (henceforth the cognitive or secondary cell). One factor that makes it challenging to apply the idea of interference alignment to a cellular system is the fact that if we align the interference signals on one of the users they may not be aligned at the rest of the users in the same cell.

Each mobile user experiences two kinds of interference: 1) the interference caused by the other cell or the *inter-cell* interference, 2) the interference caused by the message intended for the other users within its own cell or the *intra-cell* interference. In this work we assume that the primary BS only has channel state information (CSI) knowledge of the channel realizations between itself and the primary users, while the cognitive BS is assumed to have full CSI knowledge including the CSI of the primary base stations to its users. Also, we want to make sure few changes are made to the communication scheme of the primary cell.

In our work, using the cognitive knowledge at the secondary base station, we cancel the intra-cell interference on the primary users by interference cancellation. In addition, the inter-cell interference from the primary base station to the secondary users is canceled using dirty paper coding (DPC) [6]. Finally, we derive an outer-bound on the DoF region and show that our proposed scheme is optimal for a special case when the cognitive BS is transmitting at the maximum possible DoF to its users.

The rest of the paper is organized as follows: Section II introduces the system model and mathematical formulation. In Section III, we explain our proposed signaling scheme in detail, the outer bounds are studied in Section IV, and finally the paper is concluded in Section V.

II. System model

We consider a cellular system with one primary and another secondary base stations denoted by P and S. Each of the base stations serve 2 mobile users in their cells: P_1, P_2 in the primary cell and S_1, S_2 in the secondary cell.

Let us denote the messages intended for primary users by W_{P_1}, W_{P_2}, and for the secondary users by W_{S_1}, W_{S_2} respectively. The total power available at the base stations is denoted by ρ. Rates of $R_{P_i}(\rho)$, $R_{S_j}(\rho)$ are said to be achievable

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with power ρ if there exists a coding scheme to reliably communicate messages of sizes $|W_{P_i}(\rho)| = 2^{nR_{P_i}(\rho)}$ and $|W_{S_j}(\rho)| = 2^{nR_{S_j}(\rho)}$ where n is the number of channel uses. The set of all achievable rate tuples at power ρ is denoted by $\mathcal{C}(\rho)$. Throughout the paper it is assumed that the secondary base station non-causally knows the messages intended for the primary users i.e., at S, $W_{P_1}(\rho), W_{P_2}(\rho)$ are known prior to transmission.

Following the notation introduced in [7], the degrees of freedom region $\mathcal{D}$ is defined as

$$\begin{aligned}
&(d_{P_1}, d_{P_2}, d_{S_1}, d_{S_2}) \in \mathbb{R}^4,\\
&\text{for all } (\alpha_{P_1}, \alpha_{P_2}, \alpha_{S_1}, \alpha_{S_2}) \in \mathbb{R}^4\\
&\text{such that } \sum_{i=1}^{2}\sum_{j=1}^{2}(\alpha_{P_i} d_{P_i} + \alpha_{S_j} d_{S_j}) \leq\\
&\lim\sup_{\rho\to\infty} \frac{1}{\log(\rho)}\Big(\sup_{\mathcal{C}(\rho)}\sum_{i=1}^{2}\sum_{j=1}^{2}(\alpha_{P_i} R_{P_i}(\rho) + \alpha_{S_j} R_{S_j}(\rho))\Big).
\end{aligned}$$

The maximum sum of degrees of freedom is defined as:

$$d_m = \max_{(d_{P_1}, d_{P_2}, d_{S_1}, d_{S_2})\in\mathcal{D}} (d_{P_1} + d_{P_2} + d_{S_1} + d_{S_2}) \tag{1}$$

Also the sum DoF in the primary and secondary cells are defined respectively by:

$$d_P = \max_{(d_{P_1}, d_{P_2})\in\mathcal{D}} (d_{P_1} + d_{P_2}), \quad d_S = \max_{(d_{S_1}, d_{S_2})\in\mathcal{D}} (d_{S_1} + d_{S_2})$$

The users are assumed to be equipped with multiple antennas. Let m_P and m_S denote the number of antennas at the primary and secondary base stations and let n_P and n_S denote the number of antennas at the primary and secondary users respectively. Throughout the paper we assume:

$$n_P \leq \min(m_P, m_S) \tag{2}$$

Basically, (2) means that the number of antennas on either of the base stations, is not less than that of the primary mobile users. Using time or frequency expansions [1] we can generate M extra dimensions. Denote the number of available dimensions on each node by: $M_P = M \times m_P$, $M_S = M \times m_S$, $N_P = M \times n_P$ and $N_S = M \times n_S$ respectively.

The received signal at user P_i is equal to:

$$\mathbf{y}_{P_i} = \mathbf{H}_{P_i}\mathbf{x}_P + \mathbf{H}'_{P_i}\mathbf{x}_S + \mathbf{z}_{P_i}, \tag{3}$$

where $\mathbf{H}$'s represent the MIMO channel coefficients to the users and $\mathbf{H}'$'s are the MIMO channel from the cognitive base station. For both, $\mathbf{H}$ and $\mathbf{H}'$ the subscript denotes the receiver. For example, $\mathbf{H}_{P_i}$ which is a matrix of size $N_P \times M_P$ and $\mathbf{H}'_{P_i}$ which is $N_P \times M_S$ are the channels from the primary and secondary base stations to the i-th primary user respectively. $(\mathbf{x}_P)_{M_P\times 1}$ and $(\mathbf{x}_S)_{M_S\times 1}$ denote the signals transmitted from the primary and secondary base stations respectively and $\mathbf{z}$'s are the zero mean unit variance i.i.d. additive white Gaussian noise.

[1]Through multiple fading blocks and/or OFDM subcarriers

Similarly, the signal received at the j-th secondary mobile user will be:

$$\mathbf{y}_{S_j} = \mathbf{H}_{S_j}\mathbf{x}_S + \mathbf{H}'_{S_j}\mathbf{x}_P + \mathbf{z}_{S_j} \tag{4}$$

In this paper, we assume that the channel coefficients are drawn independently from a continuous distribution and thus the channel matrices are full rank almost surely. Also, we assume that the cognitive base station has perfect CSI knowledge; while the primary base station only has its own cell's CSI information. The users are assumed to be able to estimate their receive CSI using training sequences.

III. Main Result

In this section, we explain our proposed signaling scheme in detail and present an achievable region for the DoF.

Theorem 1: Let $\mathfrak{D}_{in}$ be the convex hull of the 4-tuples $(d_{P_1}, d_{P_2}, d_{S_1}, d_{S_2})$ with $Md_{P_1} = \bar{d}_{P_1}$, $Md_{P_2} = \bar{d}_{P_2}$, $Md_{S_1} = \bar{d}_{S_1}$ and $Md_{S_2} = \bar{d}_{S_2}$, for $\bar{d}_{P_1}, \bar{d}_{P_2}, \bar{d}_{S_1}, \bar{d}_{S_2} \in \mathbb{Z}^+ \cup \{0\}$, satisfying the following inequalities for the (M_P, M_S, N_P, N_S) cognitive cellular system.

$$\begin{aligned}
\bar{d}_{P_1} &\leq M_P\\
\bar{d}_{P_2} &\leq M_P\\
\bar{d}_{P_1} + \bar{d}_{S_1} + \bar{d}_{S_2} &\leq N_P\\
\bar{d}_{P_2} + \bar{d}_{S_1} + \bar{d}_{S_2} &\leq N_P\\
\bar{d}_{S_1} + \bar{d}_{S_2} &\leq M_S\\
\bar{d}_{S_1} &\leq N_S\\
\bar{d}_{S_2} &\leq N_S
\end{aligned} \tag{5}$$

Then, $\mathfrak{D}_{in} \subseteq \mathcal{D}$.

Proof: Let $Z = (M_P - N_P)^+$ which is equal to the dimension of the null space of the channel between the primary BS and the primary users. The primary base station picks $\mathbf{v}_1^{P_1}, ..., \mathbf{v}_Z^{P_1}$ and $\mathbf{v}_1^{P_2}, ..., \mathbf{v}_Z^{P_2}$ in the null space of $\mathbf{H}_{P_2}$ and $\mathbf{H}_{P_1}$ respectively. That is for $1 \leq l \leq Z$, $\mathbf{H}_{P_2}\mathbf{v}_l^{P_1} = \mathbf{H}_{P_1}\mathbf{v}_l^{P_2} = 0$. The rest of $\bar{d}_{P_1} + \bar{d}_{P_2} - 2Z$ vectors of size $(M_P \times 1)$; $\mathbf{v}_{Z+1}^{P_1}, ..., \mathbf{v}_{\bar{d}_{P_1}}^{P_1}, \mathbf{v}_{Z+1}^{P_2}, ..., \mathbf{v}_{\bar{d}_{P_2}}^{P_2}$ are picked randomly with elements chosen according to an isotropic distribution. The transmitted signal at the primary base station is:

$$\mathbf{x}_P = \sum_{i=1}^{\bar{d}_{P_1}} \mathbf{v}_i^{P_1} x_i^{P_1} + \sum_{j=1}^{\bar{d}_{P_2}} \mathbf{v}_j^{P_2} x_j^{P_2}$$

where $x_i^{P_1}$ and $x_j^{P_2}$'s are the data streams to users P_1 and P_2 respectively using a Gaussian codebook.

The transmission scheme from the secondary BS has two goals:

1) To cancel the intra-cell interference on the primary users.

2) To reliably transmit data to its own users. That latter goal is achieved by using dirty paper coding and zero forcing at the transmitter. More specifically, the signal transmitted from

the secondary BS is equal to:

$$\mathbf{x}_S = \sum_{g=1}^{\bar{d}_{S_1}} \mathbf{v}_g^{S_1} \hat{x}_g^{S_1} + \sum_{h=1}^{\bar{d}_{S_2}} \mathbf{v}_h^{S_2} \hat{x}_h^{S_2} + \sum_{i=Z+1}^{\bar{d}_{P_1}} \bar{\mathbf{v}}_i^{P_1} x_i^{P_1} + \sum_{j=Z+1}^{\bar{d}_{P_2}} \bar{\mathbf{v}}_j^{P_2} x_j^{P_2} \quad (6)$$

First, let us explain the way the vectors $\bar{\mathbf{v}}_i^{P_1}$'s and $\bar{\mathbf{v}}_j^{P_2}$'s are selected. The goal is to zero force the *intra-cell* interference on each primary user by transmitting the unintended data from the cognitive base station on the opposite direction. To this end we pick:

$$\begin{aligned} \bar{\mathbf{v}}_i^{P_1} &= -{\mathbf{H}'_{P_2}}^T (\mathbf{H}'_{P_2}(\mathbf{H}'_{P_2})^T)^{-1} \mathbf{H}_{P_2} \mathbf{v}_i^{P_1}, \\ \bar{\mathbf{v}}_j^{P_2} &= -{\mathbf{H}'_{P_1}}^T (\mathbf{H}'_{P_1}(\mathbf{H}'_{P_1})^T)^{-1} \mathbf{H}_{P_1} \mathbf{v}_j^{P_2}, \\ for \quad & i = (Z+1), \cdots \bar{d}_{P_1}, \; j = (Z+1), \cdots \bar{d}_{P_2} \end{aligned}$$

Note that the matrices $(\mathbf{H}'_{P_2}(\mathbf{H}'_{P_2})^T)$ and $(\mathbf{H}'_{P_1}(\mathbf{H}'_{P_1})^T)$ are invertible when $N_P \leq M_S$.

It remains to specify the transmission scheme from the secondary BS to the secondary users. The beamforming vectors of the secondary BS to the secondary users, are determined using the common zero forcing approach for the MIMO broadcast channel. That is, let $\mathbf{H}_{S_1} = [(\mathbf{h}_{S_1}^1)^T; (\mathbf{h}_{S_1}^2)^T; ...; (\mathbf{h}_{S_1}^{N_S})^T]$ and $\mathbf{H}_{S_2} = [(\mathbf{h}_{S_2}^1)^T; (\mathbf{h}_{S_2}^2)^T; ...; (\mathbf{h}_{S_2}^{N_S})^T]$ where each $(\mathbf{h}_{S_k}^i)$ is an $M_S \times 1$ vector corresponding a row in the channel matrices of S_1 and S_2 to S. Without loss of generality we pick the first $\bar{d}_{S_1}$ and the first $\bar{d}_{S_2}$ rows of $\mathbf{H}_{S_1}$ and $\mathbf{H}_{S_2}$. Next, the beamforming vectors $\mathbf{v}_1^{S1}, ..., \mathbf{v}_{\bar{d}_{S_1}}^{S1}$ and $\mathbf{v}_1^{S2}, ..., \mathbf{v}_{\bar{d}_{S_2}}^{S2}$ are picked as the basis of the null space spanned by the rest of channel vectors [9]. This way by zero forcing at the transmitter we produce $\bar{d}_{S_1}$ and $\bar{d}_{S_2}$ parallel channels to S_1 and S_2 respectively. Moreover, we apply dirty paper coding in the way explained below. Consider the mobile user S_1 first. The following term is completely *known* at S

$$\mathbf{i}_{S_1} = \sum_{i=Z+1}^{\bar{d}_{P_1}} \mathbf{H}_{S_1} \bar{\mathbf{v}}_i^{P_1} x_i^{P_1} + \sum_{i=1}^{\bar{d}_{P_1}} \mathbf{H}'_{S_1} \mathbf{v}_i^{P_1} x_i^{P_1} + \sum_{j=Z+1}^{\bar{d}_{P_2}} \mathbf{H}_{S_2} \bar{\mathbf{v}}_j^{P_2} x_j^{P_2} + \sum_{j=1}^{\bar{d}_{P_2}} \mathbf{H}'_{S_2} \mathbf{v}_j^{P_2} x_j^{P_2} \quad (7)$$

The data stream carried by $\mathbf{v}_g^{S_1}$ is dirty paper coded against the known interference $\mathbf{i}_{S_1}$ on the dimension (antenna and time/frequency expansion) corresponding to it and $x_g^{S_1}$ will be DPC encoded in $\hat{x}_g^{S_1}$.

The achievability of $\bar{d}_{P_1}$ and $\bar{d}_{P_2}$ expanded[2] degrees of freedom depends on the number of interference free dimensions at users P_1 and P_2. Because DoF is defined asymptotically as $SNR \to \infty$, for simplicity of presentation in order to analyze the achievable DoF similar to [7] we do not include the noise vector and the signal recieved at the first primary user will be equal to:

$$\begin{aligned} \mathbf{y}_{P_1} = \; & \mathbf{H}_{P_1}\mathbf{x}_P + \mathbf{H}'_{P_1}\mathbf{x}_S = \\ & \sum_{i=1}^{Z} \mathbf{H}_{P_1} \mathbf{v}_i^{P_1} {x_i}^{P_1} + \sum_{i=Z+1}^{\bar{d}_{P_1}} \Big(\mathbf{H}_{P_1} - \mathbf{H}'_{P_1} {\mathbf{H}'_{P_2}}^T \\ & \Big(\mathbf{H}'_{P_2} (\mathbf{H}'_{P_2})^T \Big)^{-1} \mathbf{H}_{P_2} \Big) \mathbf{v}_i^{P_1} {x_i}^{P_1} + \\ & \mathbf{H}'_{P_1} \Big(\sum_{g=1}^{\bar{d}_{S_1}} \mathbf{v}_g^{S_1} \hat{x}_g^{S_1} + \sum_{h=1}^{\bar{d}_{S_2}} \mathbf{v}_h^{S_2} \hat{x}_h^{S_2} \Big) \end{aligned} \quad (8)$$

The vectors carrying $x_i^{P_1}$ will be linearly independent if:

$$\bar{d}_{P_1} \leq M_P \quad (9)$$

Without loss of generality we explain the decoding scheme for $x_{Z+1}^{P_1}$. Similar to [7] the user P_1 finds a zero forcing vector $\mathbf{u}_{Z+1}^{P_1}$ orthonormal to the space not containing $\mathbf{v}_{Z+1}^{P_1}$ and forms:

$$(\mathbf{u}_{Z+1}^{P_1})^T \mathbf{y}_{P_1} \quad (10)$$

Using the above procedure, the user P_1 can recover its data if the vectors containing data and interference be linearly independent. For this condition to hold we need to have:

$$\bar{d}_{P_1} + \bar{d}_{S_1} + \bar{d}_{S_2} \leq N_P \quad (11)$$

Similarly, we can derive the following equations at the mobile user P_2:

$$\bar{d}_{P_2} \leq M_P, \quad \bar{d}_{P_2} + \bar{d}_{S_1} + \bar{d}_{S_2} \leq N_P \quad (12)$$

The beamforming vectors to users S_1 and S_2 will all be linearly independent with probability one if:

$$\bar{d}_{S_1} + \bar{d}_{S_2} \leq M_S \quad (13)$$

In order to be able to find linearly independent zero forcing beamforming vectors for S_1 and S_2 using the scheme explained above we also need to satisfy:

$$\bar{d}_{S_1} \leq N_S \quad \bar{d}_{S_2} \leq N_S \quad (14)$$

The decoding at the secondary mobile users is carried out by dirty paper decoding. By the choice of the beamforming vectors the secondary users do not experience intra-cell interference and can recover their data by dirty paper decoding. ■

IV. OUTER BOUNDS

In this section, we present an outer bound on the achievable DoF and show that for a special case of the considered system when the cognitive base station is transmitting at its maximum DoF, the proposed signaling scheme is sum-DoF optimal.

Theorem 2: The DoF region of the considered cognitive cellular system satisfies the following bounds:

$$\begin{aligned} L_1 &: \quad d_{P_1} \leq n_P, \quad d_{P_2} \leq n_P. \\ L_2 &: \quad d_{S_1} + d_{S_2} \leq \min\{m_S, n_{S_1} + n_{S_2}\}. \\ L_3 &: \quad d_{S_1} + d_{S_2} + d_{P_i} \leq \max\{m_S, n_P\}, i = 1, 2. \end{aligned} \quad (15)$$

[2]multiplied by M which is equal to time/frequency expansion

Proof: The bound L_1 follows from the outer-bounds on the point to point MIMO channel and the fact the it cannot exceed the number of receive antennas.
L_2 follows by assuming full cooperation between the mobile users of the secondary base station and assuming they perfectly know the interference caused by the primary base station. This cannot reduce the DoF region and L_2 follows from the outer-bounds on the DoF of the point to point MIMO channel as well.
To establish L_3, we first let $d_{P_l} = 0$ for $l \neq i$ to get a bound on d_{P_i} and assume full cooperation at the secondary base station. This reduces the problem to an interference channel with a cognitive transmitter. The bound on L_3 follows by using the outer bounds on cognitive MIMO interference channel [10] which follows from the genie aided bound on the sum DoF of MIMO interference channel in [8]. The proof still follows assuming a cognitive transmitter, for completeness however we present the proof in Appendix A. The key step in [8] is the outer-bound on the sum DoF of a MIMO MAC channel. It should be noted that the primary base station does not have any CSI information from the cognitive BS, but this information can be provided to P without decreasing the DoF region. In addition, the outer-bounds on the degrees of freedom for the point to point and multiple access channel are the same with and without transmit CSI [4]. ■

Now, let us consider the system with $m_P = m_S = n_P = 2k+1$ and $n_S = k$ for an integer $k \geq 1$. Using the achievable strategy, the DoF region of $(1, 1, k, k)$ is achievable for this system. For the considered system the maximum sum DoF of the cognitive cell is equal to $2k$, which is also achieved by the proposed signaling scheme.
If we assume the cognitive cell does not loose any of its DoF by helping the primary cell and applying the bound L_3, we arrive at $d_{P_i} \leq 1$. By the proposed signaling scheme, we achieve $d_{P_i} = 1$ which shows that when the cognitive cell is communicating at its maximum sum DoF our proposed scheme achieves the optimum DoF region.
Also, consider the case when cognitive message sharing is not possible. If we assume all the users in the primary and secondary cell fully cooperate with all the users in their cell we will have a

$$\{M_1 = 2k+1, M_2 = 2k+1, N_1 = 2(2k+1), N_2 = 2k\}$$

MIMO interference channel where M_1, M_2 denote the number of dimensions on the first and second transmitter and N_1, N_2 are the number of dimensions on the first and second receiver respectively. This cooperation cannot reduce the DoF region of the considered cellular system. Using the bound derived in [8], the maximum sum DoF of this MIMO interference channel is equal to $2k+1$ whereas our proposed scheme achieves a sum DoF of $2k+2$ which shows that the proposed scheme *strictly* outperforms the case when cognitive message sharing of the primary messages is not available to the secondary base station. Remarkably, cognition at the transmitters (having primary messages at the secondary receiver) outperforms the destination cooperation (even if the destinations of each cell are connected with infinite capacity links).

V. Conclusions

We considered downlink communication in a cognitive cellular system and proposed a novel signaling scheme based on interference alignment and dirty paper coding. The achievable degrees of freedom region under the proposed scheme was determined as well as an outerbound. We establish the tightness of outerbound under some cases. Also, we showed the significance of cognitive message sharing by establishing that the achievable sum DoF strictly outperforms that of a non-cognitive case for some cellular systems. The dual MAC problem and extension to more number of mobile users and/or cells under different cases of cognition and CSI availability are under consideration for the journal version of this work.

Appendix A
Proof of the Outer bound L_3

In order to derive the bound L_3, first we assume that the secondary users fully cooperate. In this case we can "lump" the users S_1 and S_2 into one user $\mathcal{S}$ with $2n_S$ antennas. Second, we also let the primary messages be given to $\mathcal{S}$ by a genie. The above assumptions cannot reduce the achievable DoF region. Consider the system with transmitter P, S and receivers $P_1, \mathcal{S}$. The signals received at P_1 ans $\mathcal{S}$ are:

$$\begin{aligned} \mathbf{y}_{P_1} &= \mathbf{H}_{P_1}\mathbf{x}_{P_1} + \mathbf{H}'_{P_1}\mathbf{x}_S + \mathbf{z}_{P_1}, \\ \mathbf{y}_{\mathcal{S}} &= \mathbf{H}_{\mathcal{S}}\mathbf{x}_S + \mathbf{H}'_{\mathcal{S}}\mathbf{x}_P + \mathbf{z}_{\mathcal{S}}, \end{aligned} \quad (16)$$

To establish the bound we follow the steps of [8]:
1) The noise at P_1 is reduced by changing its covariance matrix to:

$$\mathbf{K}' = \mathbf{I}_{n_P} - \mathbf{H}'_{P_1}({\mathbf{H}'_{P_1}}^T\mathbf{H}'_{P_1})^{-1}{\mathbf{H}'_{P_1}}^T + \alpha\mathbf{H}'_{P_1}{\mathbf{H}'_{P_1}}^T \quad (17)$$

where $\alpha = \min(\frac{1}{\sigma^2(\mathbf{H}'_{P_1})}, \frac{1}{\sigma^2(\mathbf{H}_{\mathcal{S}})})$ in which $\sigma^2(.)$ denotes the maximum singular value of a matrix.
2) A genie provides $\mathcal{S}$ with $\mathbf{x}_{P_1}$, since $\mathbf{H}'_{\mathcal{S}}$ is known at $\mathcal{S}$, it can subtract $\mathbf{H}'_{\mathcal{S}}\mathbf{x}_{P_1}$ from its signal and get $\mathbf{y}'_{\mathcal{S}} = \mathbf{H}_{\mathcal{S}}\mathbf{x}_S + \mathbf{z}_{\mathcal{S}}$.
3) P_1 is assumed to be able to decode its message reliably thus it can also subtract $\mathbf{H}_{P_1}\mathbf{x}_P$ and arrive at $\mathbf{y}'_{P_1} = \mathbf{H}'_{P_1}\mathbf{x}_{\mathcal{S}} + \mathbf{z}'_{P_1}$.
4) Having reached the following equations:

$$\begin{aligned} \mathbf{y}'_{P_1} &= \mathbf{H}'_{P_1}\mathbf{x}_S + \mathbf{z}'_{P_1} \\ \mathbf{y}'_{\mathcal{S}} &= \mathbf{H}_{\mathcal{S}}\mathbf{x}_S + \mathbf{z}_S \end{aligned} \quad (18)$$

our aim is to show that if S can decode $\mathbf{x}_S$; P_1 will be able to decode it as well and thus the sum DoF will be less than N_P. To see this consider the singular value decomposition $\mathbf{H}_{\mathcal{S}} = \mathbf{U}_{\mathcal{S}}\mathbf{\Lambda}_{\mathcal{S}}\mathbf{V}_{\mathcal{S}}$, by multiplying $\mathbf{y}'_{\mathcal{S}}$ in $\mathbf{V}_{\mathcal{S}}^T\mathbf{\Lambda}_{\mathcal{S}}^{-1}\mathbf{U}_{\mathcal{S}}^T$ we obtain a channel with input $\mathbf{x}_S$ and a noise vector with variances $\frac{1}{\sigma^2(\mathbf{H}_{\mathcal{S}})}$.
5) $\mathbf{y}'_{P_1}$ is multiplied in $\mathbf{T} = (\mathbf{H}_{P_1}^T\mathbf{H}_{P_1})^{-1}\mathbf{H}_{P_1}$. The noise variance matrix with this operation will be $\mathbf{T}\mathbf{K}'\mathbf{T}^T$, which is straightforward to check is equal to a diagonal matrix with diagonal elements equal to α. Thus, the receiver P_1 can be made less noisy than $\mathcal{S}$ and if $\mathbf{x}(S)$ is decodable at $\mathcal{S}$ it must

be decodable, at P_1 as well.

6) Using the bound on the degrees of freedom on the MIMO channels, we can conclude that still $d_{P_1} + d_{\mathcal{S}} \leq n_{P_1}$.

7) For the case where $n_{P_1} < m_S$ and the matrix $\mathbf{H'}_{P_1}{}^T \mathbf{H'}_{P_1}$ is not invertible, similar to the argument given in [8] we can add more antennas at P_1 without hurting the DoF region and follow the steps above. In that case the sum DoF will be less than m_S.

8) Following the steps 1-7 we arrive at:

$$d_{\mathcal{S}} + d_{P_1} \leq \max(m_S, n_{P_1})$$

A similar argument can also be made for P_2 and the proof for the outerbound given in L_3 is complete.

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